J. Math. Soc. Japan
Vol. 22, No. 4, 1970

On complex manifolds with positive tangent bundles

By Shoshichi KOBAYASHI* and Takushiro OCHIAI

(Received Dec. 8, 1969)

§ 1. Introduction

Let E be a holomorphic vector bundle over a complex manifold M. To each point x of M we assign the complex projective space of complex 1-dimensional subspaces in the fibre E_x. Let $P(E)$ be the resulting fibre bundle over M with fibre $P_{r-1}(\boldsymbol{C})$, where r is the fibre dimension of E. To each point of $P(E)$ which is a complex line in a fibre of E, we assign that complex line. The resulting complex line bundle over $P(E)$ will be denoted by $L(E)$. In order to prevent any misunderstanding, we emphasize that E minus its zero section is the principal bundle associated to $L(E)$ and that, when M reduces to a point, $L(E)$ is a line bundle over $P_{r-1}(\boldsymbol{C})$ without any non-trivial holomorphic section.

For a complex line bundle we have a universally accepted notion of positivity or negativity. We say that a line bundle L over M is *semi-negative* and write $L \leqq 0$ if, for every proper holomorphic map π of a complex manifold Y into M and for every negative line bundle F over Y, the line bundle $\pi^* L^k \cdot F$ over Y is negative for every positive integer k.[1] In this paper, we say that a complex vector bundle E is *negative* and write $E<0$ if the line bundle $L(E)$ over $P(E)$ is negative. We say that E is *semi-negative* and write $E \leqq 0$ if $L(E)$ is *semi-negative* and if $L(E)^k \pi^* F$ is negative for every positive integer k and every negative line bundle F over M, where π denotes the projection $P(E) \rightarrow M$. We say that E is *positive* (resp. *semi-positive*) and write $E>0$ (resp. $E \geqq 0$) if its dual bundle E^* is negative (resp. semi-negative). It has been pointed out to us by P. Kiernan that E is negative in our sense if and only if the zero section of E has a strongly pseudo-convex neighborhood in E, i. e., weakly negative in the sense of Grauert [5]. But we omit in this paper the cumbersome adverb "weakly".

Combining results of Leray and Bott with Kodaira's vanishing theorems, we obtain vanishing theorems for positive or negative vector bundles. From these vanishing theorems and Riemann-Roch-Hirzebruch theorem we prove

* Supported partially NSF Grant GP.

1) We may adopt a weaker "semi-negativity" by considering only holomorphic bundles $\pi: Y \rightarrow M$ whose fibres are complex projective spaces.