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Multiplications in cohomology theories with coefficient maps

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Introduction

S. Araki and H. Toda [3] discussed the multiplicative structures in mod q generalized cohomology theories. The mod q (reduced) cohomology of a cohomology h was defined by $\tilde{h}^i(\ ; Z_q)=\tilde{h}^{i+2}(\ \wedge M_q)$ where M_q is a co-Moore space of type $(Z_q, 2)$. In this paper we consider the cohomology theories with stable maps of spheres as coefficients, i.e., let C_α be a mapping cone of a stable map $\alpha \in \{S^{r+k-1}, S^r\}$ then α-coefficient (reduced) cohomology of a cohomology h is defined by $\tilde{h}^i(\ ; \alpha)=\tilde{h}^{i+r+k}(\ \wedge C_\alpha)$. And we discuss the multiplicative structures in α-coefficient cohomology theories by postulating three axioms (A_1), (A_2) and (A_3) as in [3]. (A_1) and (A_3) are quite similar with these of [3], but (A_2) is not a routine generalization of that of [3] but contains an extra element. A multiplication in an α-coefficient cohomology theory satisfying (A_1), (A_2) and (A_3) is called "an admissible multiplication".

One of the important examples of α-coefficient cohomologies other than mod q theories is the case $\alpha=\eta$ (a stable class of the Hopf map $S^3 \to S^2$) and $h=KO$. In the case $\alpha=\eta$ for any h a sufficient condition for existence of admissible multiplications in $\tilde{h}(\ ; \eta)$ is obtained (Theorem 4.11). Obviously $\widetilde{KO}$-theory satisfies this condition. In this case $\widetilde{KO}(\ ; \eta)$ can be identified with the cohomology $\widetilde{KU}$ by Wood isomorphism [2], where an admissible multiplication corresponds to the multiplication in $\widetilde{KU}$ defined by tensor products (Theorem 5.3).

Some uniqueness type theorems of admissible multiplications are discussed in general, which states that admissible multiplications are in a one-to-one correspondence with elements of a group which is specific to the considered cohomology theory under the assumption that the original multiplication is commutative and associative (Corollary 2.8). In the case of $h=KO$ and $\alpha=\eta$ this group is isomorphic to Z, hence there are countably many different admissible multiplications in $\widetilde{KO}(\ ; \eta)$.

In § 1 we exhibit some elements of α-coefficient cohomology theory: reduc-