J. Math. Soc. Japan
Vol. 22, No. 4, 1970

Differential equations on convex sets

By Michael G. CRANDALL

(Received Oct. 14, 1969)

Introduction

Recent developments in the theory of semi-groups of nonlinear transformations in Banach or Hilbert spaces have sharply brought into focus the fact that these theories must be developed for semi-groups on convex sets in order to achieve their full scope. Motivated by the results of [**1**], [**6**] and [**10**], the purpose of this note is to establish existence of solutions of a Cauchy problem of the form

$$\frac{du}{dt}=g(u, t), \qquad u(0)=x, \tag{1}$$

where the function g is only defined on a set of the form $C\times[0, a]$ for some convex set C in a Banach space. The methods used are not new (see, e.g., [**3**], [**8**]), but the main result seems to have gone unnoticed and serves to clarify some of the theory of semi-groups of nonlinear transformations and the related theory of accretive mappings in Banach spaces.

Simple (but basic) existence theorems for (1) are established in Section 1. Section 2 contains applications of these results to the theory of nonlinear pseudo-contractive and accretive operators. For aesthetic reasons, applications to the semi-group theory (where one must deal with "multi-valued" mappings) are not given here.

§1. Existence and Uniqueness

The main topic of this section is existence. Uniqueness is established only in simple cases of interest. Let X be a real Banach space and C be a closed convex subset of X. We begin by establishing a local existence theorem of some generality. Denote by $B_r(x)$ the closed ball of radius r in X centered at x. Consider the set

$$K=(B_r(x)\cap C)\times[0, a]. \tag{1.1}$$

---

* The preparation of this paper was sponsored in part by the Office of Naval Research under Contract NONR 233(76). Reproduction in whole or in part is permitted for any purpose of the United States Government.