J. Math. Soc. Japan
Vol. 22, No. 2, 1970

Orbits of one-parameter groups II

(Linear group case)

By Morikuni GOTO[1)]

(Received Feb. 17, 1969)

§ 1. Introduction.

Let $\boldsymbol{R}$ denote the field of real numbers. We denote by $\mathcal{K}$ the factor group of the additive group of $\boldsymbol{R}$ modulo the subgroup composed of integers. A compact connected one-dimensional Lie group is called a *circle*. A circle is topologically isomorphic with $\mathcal{K}$. A direct product Lie group of a finite number of circles will be called a *toral group*. By a torus we shall mean the underlying analytic manifold of a toral group.

We can classify one-parameter subgroups of Lie groups topologically into three types: (1) a *closed straight line*, which is topologically isomorphic with the additive group of $\boldsymbol{R}$; (2) a circle; and (3) a non-closed one-parameter subgroup. When a one-parameter subgroup $\mathcal{X}$ is non-closed, the closure $\overline{\mathcal{X}}$ is a toral group of dimension at least two.

We let $M(n, \boldsymbol{R})$ denote the Lie algebra of all n by n matrices with real entries, and $\mathcal{GL}(n, \boldsymbol{R})$ the *general linear group*, the group of all invertible matrices in $M(n, \boldsymbol{R})$. In this paper we shall generalize the foregoing topological classification of one-parameter subgroups of $\mathcal{GL}(n, \boldsymbol{R})$ to the following form:

THEOREM 1. *Let $\mathcal{L}$ be a closed connected subgroup, and let $\mathcal{X}$ be a one-parameter subgroup of $\mathcal{GL}(n, \boldsymbol{R})$. Then an orbit of $\mathcal{X}$ in the left coset space $\mathcal{GL}(n, \boldsymbol{R})/\mathcal{L}$ is either locally compact and homeomorphic with a point, $\boldsymbol{R}$ or $\mathcal{K}$, or there exists an analytic submanifold $\mathcal{M}$ in $\mathcal{GL}(n, \boldsymbol{R})/\mathcal{L}$, which is a torus, such that the orbit can be regarded as an everywhere dense one-parameter subgroup with respect to the toral group structure of $\mathcal{M}$.*

We note here that although a locally compact one-parameter subgroup is closed (and vice versa), a locally compact orbit is not necessarily closed. Also it is to be noted that in general it is impossible to find a toral subgroup $\mathcal{T}$ of $\mathcal{GL}(n, \boldsymbol{R})$ such that an orbit of $\mathcal{T}$ coincides with the torus $\mathcal{M}$ in Theorem 1.

When $\mathcal{L}$ is a (not necessarily connected) algebraic subgroup in $\mathcal{GL}(n, \boldsymbol{R})$,

---

1) Research supported in part by NSF Grant GP 4503.