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Periods of rational forms on certain elliptic surfaces

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Introduction

In this paper we study some properties of periods of rational forms on a regular elliptic surface $B(\tau, \sigma)$ of basic type and some related problems. (For the theory of elliptic surfaces compare K. Kodaira [8] and A. Kas [6]. We mainly follow their notations.) It is known that such surfaces are suitable completions of the affine surfaces defined by

$$(1) \qquad y^2 = 4x^3 - (\tau_{4k}u^{4k} + \cdots + \tau_1 u + \tau_0)x - (u^{6k} + \sigma_{6k-1}u^{6k-1} + \cdots + \sigma_1 u + \sigma_0),$$

where we denote by (u, x, y) the variables of the three dimensional affine space $\boldsymbol{C}^3$ and by $(\tau) = (\tau_{4k}, \cdots, \tau_0)$, $(\sigma) = (\sigma_{6k-1}, \cdots, \sigma_0)$ the parameters. We understand by *a period function* $W(\tau, \sigma)$ of a rational form of the surface $B(\tau, \sigma)$ the holomorphic function of the parameters (τ, σ) of the form: $W(\tau, \sigma) = \int_{\Gamma(\tau,\sigma)} \omega(\tau, \sigma)$, where the rational form $W(\tau, \sigma)$ on $B(\tau, \sigma)$ and the homology class $\Gamma(\tau, \sigma)$ on $B(\tau, \sigma)$ satisfy conditions of continuity with respect to the parameters (τ, σ). Our chief object is to study certain properties of the period functions $W(\tau, \sigma)$.

We sketch our results briefly. After recalling some basic notions and definitions of elliptic surfaces needed below, we study in the first chapter the periods of rational 2-forms on the elliptic surface B_0: $y^2 = 4x^3 - (u^{6k} - 1)$ in details. These periods may be regarded as generalizations of Beta integrals: $\int_0^1 u^{p-1}(u-1)^{q-1}du$ in the theory of hypergeometric functions (F. Klein [7]). The study of these periods leads to power series expansions of the period functions $W(\tau, \sigma)$ at the point $(\tilde{\tau}^0, \tilde{\sigma}^0) = ((0, \cdots, 0), (0, \cdots, 0, 1))$. We derive some applications from these power series expansions: the determination of the rank of 'period maps' of individual rational forms, etc.

We also know that these period functions are complete solutions of certain partial linear differential equations $\tilde{D}$ of the second order. In the second chapter we study the structure of $\tilde{D}$ and some related problems. In § 4∼§ 5, we find a criterion of 'regularity' for systems of linear partial differential equations. This elementary criterions means merely that the solutions are 'regular' if and only if certain ordinary differential equations, which correspond to the partial equations in question, are regular in the usual sense ([c. f.] Lemma 5.1).