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Hecke polynomials of modular groups and congruence zeta functions of fibre varieties

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§ 0. Introduction.

0-1. In [HP], Y. Ihara proved that the Hecke polynomials $H_k^{(p)}(u)$ of the elliptic modular group $SL(2, Z)$ can be expressed by the congruence zeta functions of some algebraic varieties. There, he used some properties of imaginary quadratic fields and elliptic curves defined over finite fields. But in the preface of [HP], he stated that more intrinsic proof and generalization to higher level cases would be obtained by using the group

$$\Gamma_p = PL^+(2, Z^{(p)}) = \{x \in GL(2, Z^{(p)}) \mid \det x = p\text{-power}\}/\pm p\text{-powers},$$

where $Z^{(p)} = \bigcup_{n=0}^{\infty} p^{-n}Z \subset Q$. We shall carry out this program in this paper.

0-2. Let p be a prime number, $\Gamma = PSL(2, Z^{(p)})$ and Δ be its subgroup of finite index. We shall define the Hecke polynomial of Δ and study it in this paper. We shall treat only subgroups of $PSL(2, Z^{(p)})$, which is a subgroup of $PL^+(2, Z^{(p)})$ of index 2. This restriction simplifies the calculations and notations to a fair degree. Moreover, this restriction makes no difference if we are interested only in the absolute values of the zeros of the Hecke polynomials (cf. § 1, Remark 2). But, of course, we can obtain similar results in the general case.

In the first section, we define the Hecke operators $T_k(\Delta, m)$ which act on the space of the cusp forms of weight k ($k = 2, 4, \cdots$) with respect to the Fuchsian group $\Delta^0 = \Delta \cap PSL(2, Z)$. If $\Delta = PSL(2, Z^{(p)})$, then our operators coincide with the well-known Hecke operators $T_k(p^{2m})$ which were studied by Hecke. We prove a recursion formula of $T_k(\Delta, m)$ and consequently obtain the following equality;

$$\sum_{m=1}^{\infty} \frac{1}{p^{ms}} T_k(\Delta, m) = \{1 - p^{2(k-1)}p^{-s}I\}/\{I - (T_k(\Delta, 1) - p^{k-1}I)p^{-s} + p^{2(k-1)}Ip^{-2s}\},$$

where I denotes the identity operator. So, we define the Hecke polynomial by

$$H_k(\Delta\,;\, u) = \det\{I - (T_k(\Delta, 1) - p^{k-1}I)u + p^{2(k-1)}Iu^2\},$$