J. Math. Soc. Japan
Vol. 21, No. 4, 1969

On injective modules

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(Received Jan. 23, 1969)
(Revised April 4, 1969)

In [3] C. Faith and E. A. Walker gave a characterization of a left artinian ring in terms of module theories. That is, a ring B is left artinian if and only if every injective left B-module is a direct sum of injective hulls of simple left B-modules. Under the assumption that a ring B is commutative, P. Vámos investigated in [9] some conditions for B to be locally artinian. One part of this paper is concerned with these results, that is, we give some conditions for a commutative ring R such that there exists a finitely generated injective R-module. The details are the following: Let R be a commutative ring with the noetherian total quotient ring. Then we have the followings: (1) There is a torsion-free and finitely generated injective R-module if and only if there exists a maximal ideal $\mathfrak{M}$ in R such that $R_{\mathfrak{M}}$ is an artinian local ring (Theorem 1). (2) There is a cyclic injective R-module if and only if there exists a maximal ideal $\mathfrak{M}$ in R such that $R_{\mathfrak{M}}$ is a self-injective artinian local ring (Theorem 3).

The other part of this paper is concerned with the ring property of an injective hull of a commutative ring. Let B be a ring and let $E_B(B)$ be an injective hull of B. Then we call $E_B(B)$ a *B-algebra* only when $E_B(B)$ (identifying B with its canonical image in $E_B(B)$) has a B-algebra structure and the multiplication between an element of B and an element of $E_B(B)$ as a B-algebra coincides with the multiplication as a B-module. In [7] B. L. Osofsky gave an example of a non commutative ring B whose injective hull is not a B-algebra. Even when a ring is commutative, such a ring exists (Theorem 4). Now we give here a necessary and sufficient condition for a commutative ring of special type such that its injective hull is an R-algebra. The result is the following: Let R be a commutative ring whose total quotient ring is artinian. Then an injective hull of R can be made into an R-algebra if and only if the total quotient ring of R is a self-injective ring (Theorem 6).

In this paper we assume always that a ring is commutative and has a unit element and a module is unitary. Let R be a ring. We denote an injective hull of an R-module M by $E_R(M)$, the set of all the regular elements (=non zero-divisors) in R by $S(R)$, and the total quotient ring of R by $Q(R)$.