J. Math. Soc. Japan
Vol. 21, No. 3, 1969

A generalization of F. Schur's theorem

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(Received Feb. 13, 1969)

The following theorem, due to F. Schur, is well-known:

THEOREM A. *Let M be a Riemannian manifold with* $\dim M \geqq 3$. *If the sectional curvature K of M is constant at each point of M, then K is actually constant on M.*

There are several other theorems of this type; we mention a few of them.

THEOREM B. *Let M be an Einstein manifold, that is, assume the Ricci curvature of M is a scalar multiple λ of the metric tensor of M. If* $\dim M \geqq 3$, *then λ is constant.*

THEOREM C (Thorpe [2]). *Let M be a Riemannian manifold with* $\dim M \geqq 2p+1$. *If the $2p$th sectional curvature γ_{2p} is constant at each point of M, then γ_{2p} is constant on M.*

THEOREM D. *Let M be a Kähler manifold with* $\dim M \geqq 4$. *If the holomorphic sectional curvature K_h is pointwise constant, then it is actually constant.*

THEOREM E (M. Berger, unpublished). *Let M be a Riemannian manifold with metric tensor g_{ij} and Riemann curvature tensor R_{ijkl}. Suppose*

$$\sum_{i,j,k} R_{ijks}R^{ijkt} = \lambda g_{st} .$$

If $\dim M \geqq 5$, *then λ is constant.*

In this paper we prove a result (theorem 2) which includes theorems A, B, C, and D as special cases. Although theorem E is not a consequence of theorem 2, it almost is, in the sense that it would be if a slightly different contraction were used.

We shall use the notation of [1]. Recall that a *double form* of type (p, q) is a function $\omega: \mathfrak{X}(M)^{p+q} \rightarrow \mathcal{F}(M)$ which is skew-symmetric in the first p variables and also in the last q variables. Here, as usual, $\mathfrak{X}(M)$ denotes the Lie algebra of vector fields on the C^∞ manifold M and $\mathcal{F}(M)$ the ring of C^∞ real valued functions on M. We write $\omega(X_1, \cdots, X_p)(Y_1, \cdots, Y_q)$ for the value of ω on $X_1, \cdots, X_p, Y_1, \cdots, Y_q$. If $p=q$ and

$$\omega(X_1, \cdots, X_p)(Y_1, \cdots, Y_p) = \omega(Y_1, \cdots, Y_p)(X_1, \cdots, X_p)$$

for

$$X_1, \cdots, X_p, Y_1, \cdots, Y_p \in \mathfrak{X}(M) ,$$