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Class numbers of definite Hermitian forms

By Kenichi IYANAGA

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§ 0. Introduction.

G. Shimura has studied about the arithmetic of Hermitian forms over a quadratic extension K of an algebraic number field of finite degree k (cf. [7]). He proved that the special unitary class number of an indefinite Hermitian form is 1, and that the unitary class number of such a form can be described in terms of the class number of K.

Our purpose is to determine the (special unitary, and unitary) class numbers of definite Hermitian forms. In general, this problem does not seem to be easy. We have been able to get only very partial solutions.

M. Kneser has developed a method to determine the class numbers of definite quadratic forms in small numbers of variables with simple discriminants [4]. His method can be applied to determine the class numbers of certain families of Hermitian forms.

In particular, we get the following:

Let $K=\boldsymbol{Q}(\sqrt{-1})$, $k=\boldsymbol{Q}$. Take a vector space V over K with the bases $v_1, \cdots, v_n$. Let H be the Hermitian form determined by

$$H(v_i, v_j)=\delta_{ij}.$$

Let

$$L=\sum_{i=1}^{n} \boldsymbol{Z}[\sqrt{-1}]v_i$$

be the lattice in V.

In this case, it turns out that the unitary class number c_n of the lattice L and the special unitary class number c_n^1 of L, have the relation: $c_n \leqq c_n^1 \leqq (n, 4)\, c_n$ where $(n, 4)$ is the G. C. D. of n and 4; $c_n^1=1$ if $c_n=1$.

Moreover, we have $c_n>1$ if $n\geqq 5$, and $c_1=c_2=c_3=c_4=1$, $c_5=2$, $c_6=3$ and $c_7=4$. (In fact we can directly apply the results of M. Kneser [4] to get $c_i=1$ for $i=1, 2, 3, 4$.) $c_6^1=3$ or 4.

In § 1, we investigate the relations of the unitary class numbers and the special unitary class numbers. We will discuss about Kneser's method in § 2. In the last section, we will calculate the above c_n's.

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