J. Math. Soc. Japan
Vol. 21, No. 2, 1969

On the second cohomology groups of the fundamental groups of simple algebraic groups over perfect fields

By Takashi TASAKA

(Received June 17, 1968)

Introduction.

In this paper, we determine the first and second cohomology groups of the following tori: $\boldsymbol{G}_m$, $R_{K/k}(\boldsymbol{G}_m)$, and some tori associated with $R_{K/k}(\boldsymbol{G}_m)$ (U and V defined in § 2) and discuss relations between them. As an application, we also determine $H^2(k, Z)$, where Z is the center of a simply connected simple algebraic group F defined over a perfect field k. Since any simply connected simple algebraic group F defined over k is obtained by an inner twist from a certain quasi-split simple algebraic group F_1 defined over k, in order to determine $H^2(k, Z)$, it suffices to determine $H^2(k, Z_1)$, where Z_1 is the center of F_1.

In n°1, we state some lemmas which are well-known. In n°2, we determine the cohomology groups of some special tori, applying the lemmas to the case $M=k_s^*$, where k_s is the separable closure of k. In n°3 and n°4, we determine $H^2(k, Z)$ and define an H^2-invariant of a k-form of a simple algebraic group. N°5 has a nature of an appendix which will explain in a certain sense the meaning of the table obtained in n°3. Let K be a separable quadratic extension of an arbitrary field k. We prove that a central simple algebra B over K has an anti-automorphism over k if and only if $\beta+\bar{\beta}=0$, where β is the class of B in the Brauer group $B(K)$ of K. We also prove that B has an involution over k if and only if $c(\beta)=0$, where c is the corestriction of $B(K)$ into $B(k)$.

The author would like to express his hearty thanks to Dr. T. Kondo and Dr. H. Hijikata who have read his first manuscript critically and have given him useful suggestions.

§ 1. Preliminaries.

Let $\mathfrak{g}$ be an arbitrary group and $\mathfrak{h}$ be its subgroup of finite index n. Put $\mathfrak{g}=\bigcup_{i=1}^{n} g_i\mathfrak{h}$, with $g_1=1$. Putting