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Fractional powers of operators, IV
Potential operators

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A closed linear operator A in a Banach space X is said to be non-negative if $(0, \infty)$ is contained in the resolvent set of $-A$ and if $\lambda(\lambda+A)^{-1}$ is uniformly bounded for $0<\lambda<\infty$. This is a short supplement to the third paper of the author's series on fractional powers of non-negative operators A and mainly concerned with the potential operator associated with A, which is by definition the inverse A_{-}^{-1} of the restriction A_{-} of A to the closure $\overline{R(A)}$.

A typical result is the Abel and the Cesàro (the Cauchy) convergence of the integral formula

$$A_{-}^{-1}x=\int_0^{\infty}T_s x\, ds$$

when $-A$ is the infinitesimal generator of a bounded continuous (analytic resp.) semi-group T_t. A related integral formula of A_{-}^{α} with $\operatorname{Re}\alpha<0$ is also investigated.

§1. Potential operators.

Suppose that $-A$ generates a bounded continuous semi-group T_t in a Banach space X. Then for $\lambda>0$ we have

$$(\lambda+A)^{-1}x=\int_0^{\infty}e^{-\lambda s}T_s x\, ds\,, \qquad x\in X\,. \tag{1.1}$$

Letting $\lambda\to 0$, we may expect that

$$A^{-1}x=\int_0^{\infty}T_s x\, ds\,, \qquad x\in D(A^{-1})\,. \tag{1.2}$$

Of course, this is not true in general, for A need not be even one-to-one.

Yosida [4] proves, however, that A has a densely defined inverse A^{-1} if and only if

$$\lambda(\lambda+A)^{-1}x\to 0 \qquad \text{as } \lambda\to 0 \tag{1.3}$$

for all $x\in X$ and that if this is the case, then the potential operator A^{-1} is