J. Math. Soc. Japan
Vol. 21, No. 1, 1969

On restricted roots of semi-simple algebraic groups*)

By Doris J. SCHATTSCHNEIDER

(Received March 5, 1968)

§ 0. Introduction and notation.

In this paper, we are interested in re-examining, under more general assumptions, some of the recent work of Satake, Tits and Borel concerning restricted roots of semi-simple algebraic groups, and the Weyl group associated to these roots ([1], [4]). Their work concentrates on the study of the system of k-roots of a connected semi-simple (or reductive) algebraic group G defined over a ground field k, and hence the Galois group $G(K/k)$, where K is a splitting field for a maximal torus of G defined over k, plays an important role. The initial question which led to this paper was "what is the importance of the maximal k-trivial torus and the Galois group in this study?" That is, are there more general assumptions on a subtorus of G under which much of the theory holds true, and can the Galois group be replaced by a more general automorphism group of the root system of G?

We will show that both of these questions have affirmative answers, and obtain necessary and sufficient conditions for a large class of tori (called *admissible* tori) to induce sets of restricted roots which possess many of the properties of k-roots. Since maximal k-trivial tori are a special case of all the admissible tori we consider, many of our theorems yield properties of maximal k-trivial tori. Only a few of these properties are not proved in [1], [4]; however, it is hoped that our method of proof indicates that many of these properties are equivalent, and depend on a minimum set of assumptions.

Throughout the paper, we will use the following standard notation (patterned after that in [4]).

G: a connected reductive algebraic group, (assumed semi-simple in § 2–§ 5)

T: a fixed maximal torus of G

$X=X(T)$: the group of rational characters of T

$\mathfrak{r}$: the root system of G with respect to T

W: the Weyl group of $\mathfrak{r}$

w_α: the element of W which is the reflection with respect to $\alpha \in \mathfrak{r}$.

We will denote by G_a and G_m the one-dimensional additive and multiplicative

*) This work was partially supported by NSF grant GP 6539.