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Finite simple groups with short chains of subgroups

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§ 1. Introduction.

Let G be a finite group. Let $G=G_0>G_1>\cdots>G_n=1$ be a chain of subgroups of G, where G_i is a proper subgroup of G_{i-1} $(1\leqq i\leqq n)$. Then we say that the chain has length n. For a fixed group G we denote by $l(G)$ the maximal number of n, where the chain ranges all possible ones. We call $l(G)$ the length of G.

In his papers [**16**], [**17**] Z. Janko has proved that if G is a finite non-abelian simple group whose length $l(G)$ is at most four, then G is isomorphic to $PSL(2, p)$ for some prime p. Moreover, he has proved that if $l(G)$ is at most five, then G is isomorphic to $PSL(2, q)$ for some prime power q.

In this note we shall prove the following extention of the above theorem of Z. Janko.

THEOREM 1. *Let G be a finite non-abelian simple group whose length $l(G)$ is at most seven. Then G is isomorphic to one of the following groups:*

(1) *$PSL(2, q)$, for a suitable prime power q,*

(2) *$PSU(3, 3^2)=U_3(3)$, $PSU(3, 5^2)=U_3(5)$,*

(3) *A_7 the alternating group of degree seven,*

(4) *M_{11} the Mathieu group of degree* 11,

(5) *J the Janko group of order* 175560 [**18**].

The proof of the theorem proceeds by considering the structure of centralizers of involutions and the Sylow 2-subgroups of G. Assuming that there exists a simple group G which is not isomorphic to any one of the groups mentioned above, we are able to prove that the order of the Sylow 2-subgroups of G has to be at least 2^7, which is clearly contrary to our hypothesis $l(G)\leqq 7$.

J. G. Thompson has conjectured in [**23**] that if the Sylow 2-subgroup S of a finite simple group G is maximal in G, then S should be dihedral. We shall prove that this conjecture is true if the order of S is less than or equal to 64 (Lemma 7).

In the last section of this note, we shall prove some elementary lemmas on the Sylow 2-subgroups of finite groups. For instance we shall show the following lemma.

LEMMA. *If a Sylow 2-subgroup S of a finite group G is generalized*