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Well-orderings and finite quantifiers

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§ 0. **Introduction.** That the class **W** of all (non-empty) well-orderings cannot be characterized using (finite) first-order sentences is a well known result. Almost as well known is that **W** can be characterized by an infinitely long sentence involving conjunctions of countably many formulas and quantifications over countable sequences of individual variables (cf. [8]). In [4] and [5] it is shown that in order to characterize **W** in an infinitary first-order language quantifications over infinitely many individual variables are essential. The aim of this paper is to determine how much can we express, concerning well-orderings, in infinitary languages whose only non-logical constant is a binary relation symbol and which allow the conjunction/disjunction of infinitely many formulas but whose quantifiers bind single individual variables. The results in this note are obtained by an elimination of quantifiers, that is we determine a certain class of sentences, which for lack of a better name we shall call "sentences in normal form" or simply "normal sentences", such that any other sentence is equivalent (as far as well-orderings are concerned) to a disjunction of normal sentences. The method of carrying out the elimination of quantifiers is essentially an extension of the combination of the methods used by Ehrenfeucht [2] and Mostowski/Tarski [6] for the finite language.

§ 1. **The language $\mathbf{L}_{\alpha\omega}$.** Var_α is the set of individual variables of $\mathbf{L}_{\alpha\omega}$ and $\mathrm{Var}_\alpha = \{v_\mu : \mu < \alpha\}$. The atomic formulas of $\mathbf{L}_{\alpha\omega}$ are the expressions of the form: $x \doteq y$ and $x < y$ where x and y are individual variables. The set of formulas of $\mathbf{L}_{\alpha\omega}$ is the least set S which includes all the atomic formulas and such that:

(a) if $\theta \in S$, then the negation of θ, $\neg\theta$, is also a member of S,

(b) if $X = \{\theta_i : i \in I\} \subseteq S$ and $|X| < \alpha$, then both the conjunction of X, $\bigwedge X$ (also written $\bigwedge_{i\in I}\theta_i$) and the disjunction of X, $\bigvee X$ (or $\bigvee_{i\in I}\theta_i$) are also members of S,

(c) if $\theta \in S$ and $x \in \mathrm{Var}_\alpha$, then both the universal quantification of θ, $\forall x\theta$ and the existential quantification of θ, $\exists x\theta$, are members of S.

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