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Some classes of semi-groups of nonlinear transformations and their generators[1]

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1. Introduction. Suppose the linear space X is a Banach space under the norm $\| \|$, X_0 is a linear subspace of X, not necessarily $\| \|$-closed, K is a positive cone in X_0, and $\| \|_0$ is a semi-norm on X_0. We do not suppose that $\| \|_0$ is $\| \|$-continuous. Let $S=\{x \in K : \|x\|_0 \leqq 1\}$, and $\rho(x, y)=\|x-y\|$ for x, y in S. We suppose that (S, ρ) is a complete metric space and consider semi-groups of transformations from S into S. This covers a variety of settings. If $K=X_0$, and $\| \|_0$ is a norm on X_0, then (S, ρ) is a *Saks space*, see [8]. If $K=X_0=X$, and $\| \|_0=\| \|$, then S is just the closed unit ball in X. If $K=X_0=X$, and $\| \|_0 \equiv 0$, then $S=X$. Some examples are given in Section 3 to show why we take S in this generality. For instance, no simpler setting seems sufficient to cover the case of a semi-group of transformations giving the solutions of a quasi-linear partial differential equation.

The theory of semi-groups of linear transformations in a Banach space is developed quite thoroughly in the treatise [5] of Hille and Phillips. Semi-groups in topological vector spaces are treated by Yosida in [10] and Komatsu in [6]. It turns out that many points of the linear theory hold true for semi-groups of nonlinear transformations, which have been dealt with by Browder [1], Neuberger [7], Segal [9], and the author [3]. The purpose here is to continue the development of a non-linear analogue to the Hille and Phillips theory. We now state enough definitions to enable us to describe the results of Section 2.

After this paper was submitted, and shortly before publication, the papers [6], [8], and [10] of Kato, Kōmura, and Oharu, respectively, came to the author's attention. These, together with Neuberger's paper [9], all consider problems similar to those considered here. This is especially true of [9] and [10]. All of these papers, however, restrict themselves to the case $S=X$.

Let Φ denote the collection of all transformations from S into S. A *semi-group of transformations in* S means a function G from $[0, \infty)$ into Φ such

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