J. Math. Soc. Japan
Vol. 20, No. 3, 1968

On the generators of non-negative contraction semi-groups in Banach lattices

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(Received Sept. 1, 1967)

1. Introduction. Let $\mathfrak{B}$ be a Banach lattice. That is, $\mathfrak{B}$ is a Banach space with the real scalar field $\boldsymbol{R}$ and a lattice at the same time and the both structures are related by the axioms: (i) If $f \geqq g$, then $f+h \geqq g+h$; (ii) If $f \geqq g$ and $a \in \boldsymbol{R}^+$ (the set of non-negative real numbers), then $af \geqq ag$; (iii) If $f \geqq g$, then $-f \leqq -g$; (iv) If $|f| \geqq |g|$, then $\|f\| \geqq \|g\|$. We use the notations

$$f \vee g = \sup\{f, g\}, \qquad f \wedge g = \inf\{f, g\},$$

$$|f| = f \vee (-f), \qquad f^+ = f \vee 0, \qquad f^- = -(f \wedge 0).$$

An element $f \geqq 0$ is called non-negative and the cone of non-negative elements is denoted by $\mathfrak{B}^+$. We call a family of linear operators $\{T_t; t \geqq 0\}$ from $\mathfrak{B}$ into $\mathfrak{B}$ an s-continuous non-negative contraction semi-group if they satisfy (i) $T_t T_s = T_{t+s}$ and $T_0 = I$ (identity); (ii) T_t is strongly continuous, i.e., $\operatorname{s-lim}_{t \to 0+} T_t f = f$[1] for each $f \in \mathfrak{B}$; (iii) T_t is a contraction, i.e., $\|T_t\| \leqq 1$; (iv) T_t is non-negative in the sense that T_t maps $\mathfrak{B}^+$ into itself. R.S. Phillips [7] characterized the generators of such semi-groups, introducing the notion of *dispersiveness*. He used a special type of Lumer's semi-inner product, that is, a mapping $s(f, g)$[2] from $\mathfrak{B} \times \mathfrak{B}$ into $\boldsymbol{R}$ which satisfies $s(f, g+h) = s(f, g) + s(f, h)$, $s(f, ag) = as(f, g)$, $|s(f, g)| \leqq \|f\| \|g\|$, $s(f, f) = \|f\|^2$, $s(f^+, f) = \|f^+\|^2$ and carries $\mathfrak{B}^+ \times \mathfrak{B}^+$ into $\boldsymbol{R}^+$. He called an operator A dispersive if $s(f^+, Af) \leqq 0$ for each $f \in \mathfrak{D}(A)$[3] and proved the following theorem: A is the generator of an s-continuous semi-group if and only if A is linear dispersive, $\mathfrak{D}(A)$ is dense and $\mathfrak{R}(\lambda - A) = \mathfrak{B}$ for some $\lambda > 0$. M. Hasegawa [3] noticed that the functional $\tau(f, g)$ defined by

$$\tau(f, g) = \lim_{\varepsilon \to 0+} \varepsilon^{-1}(\|f + \varepsilon g\| - \|f\|) \tag{1.1}$$

is useful for the characterization of the same generators. $\|f\| \tau(f, g)$ shares some properties with $s(f, g)$. Making use of $\tau'(f, g)$ defined by

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* The partial support of the Sakkokai Foundation is gratefully acknowledged.

1) s-lim denotes limit in the strong convergence.

2) The notation of Lumer and Phillips is $[g, f] = s(f, g)$.

3) The domain of A is denoted by $\mathfrak{D}(A)$ and the range by $\mathfrak{R}(A)$.