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On the divisibility of the class number in an algebraic number field

Dedicated to Professor Iyanaga on his sixties birthday

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§ 1. Introduction.

In this paper, we shall consider the divisibility of the class number of an algebraic number field K, namely, the problem to determine which number may be a factor of the ideal class number of K. The fact that the class number of K is divisible by an integer c shows that there exists a subgroup of order c in the absolute ideal class group $\boldsymbol{C}_K$ of K, and also means that there exists an unramified abelian extension field of degree c over K. Therefore, the problem to investigate which number may be a factor of the ideal class number of K is reduced to corresponding problems related to the orders of subgroups of $\boldsymbol{C}_K$, or the degrees of unramified abelian extension fields of K.

Among such subgroups and extension fields, what we know well are the ambiguous class group and the genus field. Namely, let F be an algebraic number field, and let K be a cyclic extension of finite degree n over F. Then, it is well-known that the ambiguous class number $a=a(K/F)$ with respect to K/F is of the following form[2)]:

$$a=h_F\cdot\frac{\tilde{\Pi}e(\mathfrak{p})}{n\cdot[\varepsilon:\eta]}\,, \tag{1}$$

where $\tilde{\Pi}e(\mathfrak{p})$ is the product of the ramification exponents of all the finite and infinite prime divisors in F with respect to K/F, h_F is the class number of F, and $[\varepsilon:\eta]$ is the index of the subgroup (η) of units, which are norms of numbers in K, in the group (ε) of units in F.

In this case where K/F is cyclic, we have on the other hand

$$g^*=a=h_F\cdot\frac{\tilde{\Pi}e(\mathfrak{p})}{n\cdot[\varepsilon:\eta]}\,, \tag{2}$$

since the relative genus number $g^*=g^*(K/F)$ with respect to K/F is equal to

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2) Cf. H. Yokoi [11], Lemma 4.