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On the relation for two-dimensional theta constants of level three

Dedicated to Professor Iyanaga on the occation of his 60th birthday

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Let $\{Q_{11}, Q_{12}, Q_{22}\}$ be a system of indeterminates and denote

$$\vartheta_{\mathbf{a}}(Q)=\sum_{\mathbf{m}\in\mathbf{Z}^2} Q\left(\mathbf{m}+\frac{\mathbf{a}}{3}, \mathbf{m}+\frac{\mathbf{a}}{3}\right) \qquad (\mathbf{a}=(a_1, a_2)\,;\ a_1, a_2=0, 1, -1),$$

where $Q\left(\mathbf{m}+\frac{\mathbf{a}}{3}, \mathbf{m}+\frac{\mathbf{a}}{3}\right)$ means $Q_{11}^{\left(m_1+\frac{a_1}{3}\right)^2} Q_{12}^{2\left(m_1+\frac{a_1}{3}\right)\left(m_2+\frac{a_2}{3}\right)} Q_{22}^{\left(m_2+\frac{a_2}{3}\right)^2}$. In the present note we shall give an explicit defining equation for the projective scheme $\mathrm{Proj}\,\mathbf{Z}[\vartheta_{(0,0)}(Q), \vartheta_{(1,0)}(Q), \vartheta_{(0,1)}(Q), \vartheta_{(1,1)}(Q), \vartheta_{(1,-1)}(Q)]$. The defining equation $\Delta(X_{(0,0)}, X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)})=0$ is a rather simple equation of degree ten. From this equation we can conclude the following important result:

Let ζ be a primitive cubic root of unity and Γ_0 a transformation group on $\mathbf{Q}(\zeta, \vartheta_{(0,0)}(Q), \vartheta_{(1,0)}(Q), \vartheta_{(0,1)}(Q), \vartheta_{(1,1)}(Q), \vartheta_{(1,-1)}(Q))$ consisting of all the elements

$$(\alpha, \beta)\,;\ \vartheta_{\mathbf{a}}(Q) \rightarrow \zeta^{\mathbf{a}\beta^t\alpha^t\mathbf{a}}\vartheta_{\mathbf{a}\alpha}(Q) \qquad (\mathbf{a}\in GF(3)^2),$$

where α, β are 2×2-matrices with coefficients in $GF(3)$ such that $\det\alpha' \neq 0$ and $\beta^t\alpha=\alpha^t\beta$. Then the invariant subfield of $\mathbf{Q}(\zeta, \vartheta_{(1,0)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(0,1)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(1,1)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(1,-1)}(Q)/\vartheta_{(0,0)}(Q))$ with respect to the group Γ_0 of automorphisms is the rational function field $\mathbf{Q}(\zeta, \sum_{\mathbf{a}\neq(0,0)}\vartheta_{\mathbf{a}}(Q)^3/\vartheta_{(0,0)}(Q)^3, \sum_{\mathbf{a}\neq(0,0)}\vartheta_{\mathbf{a}}(Q)^6/\vartheta_{(0,0)}(Q)^6, \vartheta_{(1,0)}(Q)\vartheta_{(0,1)}(Q)\vartheta_{(1,1)}(Q)\vartheta_{(1,-1)}(Q)/\vartheta_{(0,0)}(Q)^4)$.

§1. Canonical systems of theta constants on abstract abelian varieties.

1.1. Let **A** be an abelian variety defined over an algebraically closed field of characteristic p, where p is a prime number or zero. Let ξ be an algebraic equivalent class on **A** and X be a divisor in ξ. We denote by $\mathfrak{g}_X$ the group of all the points a in A such that $X_a \sim X$[1]. Since $\mathfrak{g}_X$ depends only the class ξ, we may denote $\mathfrak{g}_\xi$ instead of $\mathfrak{g}_X$. If $\mathfrak{g}_\xi$ is a finite group, the divisor class ξ (the divisor X) is called non-degenerate. For any prime number l

1) $X_a \sim X$ means that X is linearly equivalent to X.