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On the prime ideal theorem

Dedicated to Professor S. Iyanaga on his 60th birthday

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Let K be an algebraic number field. Let $\pi_K(x)$ be the number of the prime ideals $\mathfrak{p}$ with $N(\mathfrak{p})\leqq x$, then we have the following asymptotic formula;

$$\pi_K(x)=\int_2^x \frac{dt}{\log t}+O(x\exp(-c(\log x)^{1/2})) \tag{1}$$

(Landau [2], Satz 191). As a special case, if K is the rational number field, we have the formula;

$$\pi(x)=\sum_{p\leqq x}1=\int_2^x \frac{dt}{\log t}+O(x\exp(-c(\log x)^{1/2})), \tag{2}$$

which was first proved, in 1899, by de la Vallée Poussin. Since then, the remainder term of (2) has been improved by many authors; to obtain these improvements, the method of trigonometrical sums is very important. (Cf. Prachar [3], Titchmarsh [4].)

The purpose of this paper is to improve the remainder term of (1). Our main result is stated as follows;

MAIN THEOREM. *Let K be an algebraic number field. Let $\pi_K(x)$ be the number of the prime ideals whose norms are less than x. Then we have*

$$\pi_K(x)=\int_2^x \frac{dt}{\log t}+O\left(x\exp\left(-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right).$$

We shall begin by proving the following theorem concerning trigonometrical sums, which will be regarded as a generalization of the theorem of Vinogradov [5] and will be fundamental for the proof of Main Theorem;

THEOREM I. *Let $\mathfrak{a}$ be an ideal of K. Let $L(\mathfrak{a})$ be the set of the principal ideals divisible by $\mathfrak{a}$. Let t be a large number, and A and B two real numbers such that*

$$\exp((\log t)^{2/3})\leqq A<B\leqq 2A<2t^{6n/5},$$

where n is the degree of K. We define a trigonometrical sum $S(t;A,B)$ as follows;

$$S(t;A,B)=\sum_{\substack{\mathfrak{b}\in L(\mathfrak{a})\\ A\leqq N\mathfrak{b}<B}}\exp(2\pi it\log N(\mathfrak{b})).$$