Pluricanonical systems on algebraic surfaces of general type

Dedicated to Professor S. Iyanaga on his 60th birthday

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By a minimal non-singular algebraic surface of general type we shall mean a non-singular algebraic surface free from exceptional curves (of the first kind) of which the bigenus $P_2$ and the Chern number $c_1^2$ are both positive, where $c_1$ denote the first Chern class of the surface (see §3). Let $S$ denote a minimal non-singular algebraic surface of general type defined over the field of complex numbers and let $K$ be a canonical divisor on $S$. The number of non-singular rational curves $E$ on $S$ satisfying the equation $KE = 0$ is smaller than the second Betti number of $S$, where $KE$ denotes the intersection multiplicity of $K$ and $E$. We define $\mathcal{E}$ to be the union of all the non-singular rational curves $E$ with $KE = 0$ on $S$ and represent it as a sum: $\mathcal{E} = \sum \mathcal{E}_\nu$ of its connected components $\mathcal{E}_\nu$. Obviously $\mathcal{E}$ may be an empty set. Consider a holomorphic map $\Phi: z \to \Phi(z)$ of $S$ into a projective $n$-space $P^n$. We shall say that $\Phi$ is biholomorphic modulo $\mathcal{E}$ if and only if $\Phi$ is biholomorphic on $S - \mathcal{E}$ and $\Phi^{-1}\Phi(z) = z$ for $z \in \mathcal{E}_\nu$. For any positive integer $m$, we let $\Phi_{mK}$ denote the rational map of $S$ into $P^n$ defined by the pluri-canonical system $|mK|$, where $n = \dim |mK|$. Note that, if $|mK|$ has no base point, then $\Phi_{mK}$ is a holomorphic map. D. Mumford proved that, for every sufficiently large integer $m$, the pluri-canonical system $|mK|$ has no base point and $\Phi_{mK}$ is biholomorphic modulo $\mathcal{E}$ (see Mumford [6]; compare also Zariski [9], Matsusaka and Mumford [5]). His proof is based on results of Zariski [9] and covers the abstract case. On the other hand, it has been shown by Šafarevič [8] that $\Phi_{mK}$ is a birational map. The main purpose of this paper is to prove the following theorem:

**Theorem.** For every integer $m \geq 4$, the pluri-canonical system $|mK|$ has no base point and $\Phi_{mK}$ is a holomorphic map. For every integer $m \geq 6$, the map $\Phi_{mK}$ is biholomorphic modulo $\mathcal{E}$. 