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A remark on the principal ideal theorem

Dedicated to Professor S. Iyanaga on his sixtieth birthday

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1. The principal ideal theorem was conjectured by D. Hilbert and was proved by Ph. Furtwängler [**3**]. The proof was simplified by S. Iyanaga [**4**], H. G. Schumann (and W. Franz) [**8**] and E. Witt [**11**]. The purpose of this short note is to give a cohomology-theoretic interpretation of these proofs.

The problem can be formulated in any class formation. We use the same notations as in [**5**]. Let $\mathfrak{K}$ be the given family of fields. To each $k\in\mathfrak{K}$ an abelian group $E(k)$ is attached such that (i) for each extension K/k (k, $K\in\mathfrak{K}$) there is an injection $\varphi_{k/K}: E(k)\rightarrow E(K)$, (ii) for each Galois extension K/k (k, $K\in\mathfrak{K}$) the Galois group $G=G(K/k)$ operates on $E(K)$ and $\varphi_{k/K}E(k)=E(K)^{G}$ [1]. We assume, furthermore, that $\{E(K);\ K\in\mathfrak{K}\}$ satisfies the axioms of a class formation in [**5**].

Let $k, K, L\in\mathfrak{K}$, $k\subset K\subset L$ such that K/k and L/k are both Galois extensions, and let K/k be the maximal abelian extension in L/k. Hence the Galois group $H=G(L/K)$ is the commutator subgroup of the Galois group $G=G(L/k)$: $H=[G, G]$.

PRINCIPAL IDEAL THEOREM. *Under the above assumptions*

$$\text{(I)}\qquad \varphi_{k/K}E(k)\subset N_{H}E(L)\qquad \textit{for}\ \ H=[G, G]\,.$$

Since $H^{0}(G, E(L))\cong E(k)/N_{G}E(L)$, $H^{0}(H, E(L))\cong E(K)/N_{H}E(L)$ and the restriction mapping $\mathrm{res}_{G/H}: H^{0}(G, E(L))\rightarrow H^{0}(H, E(L))$ is the canonical mapping: $\alpha \bmod N_{G}E(L)\rightarrow\alpha \bmod N_{H}E(L)$, the above proposition (I) is equivalent to

$$\text{(I)*}\qquad \mathrm{res}_{G/H}H^{0}(G, E(L))=1\qquad \textit{for}\ \ H=[G, G]\,.$$

2. Let $\xi_{L/k}\in H^{2}(G, E(L))$ and $\xi_{L/K}\in H^{2}(H, E(L))$ be the canonical cohomology classes. By the fundamental theorem of J. Tate [**9**] there are isomorphisms: $H^{-2}(G, \boldsymbol{Z})\cong H^{0}(G, E(L))$ by $\eta_{L/k}\rightarrow\xi_{L/k}\cup\eta_{L/k}$ and $H^{-2}(H, \boldsymbol{Z})\cong H^{0}(H, E(L))$ by $\zeta_{L/K}\rightarrow\xi_{L/K}\cup\zeta_{L/K}$, where $\boldsymbol{Z}$ denotes the additive group of integers on which

1) For a G-group A we denote by A^{G} the set of all G-invariant elements of A, by $I_{G}A$ the set $\sum_{\sigma\in G}(\sigma-1)A$, and by ${}_{G}A$ the set of all $\alpha\in A$ such that $N_{G}\alpha=0$. Here N_{G} means the norm operation with respect to the group G.