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On explicit formulas for the norm residue symbol

Dedicated to Professor Shôkichi Iyanaga on his 60th birthday

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Let p be an odd prime and let $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers respectively. For each integer $n\geqq 0$, let $q_n=p^{n+1}$ and let Φ_n denote the local cyclotomic field of q_n-th roots of unity over $\boldsymbol{Q}_p$. We fix a primitive q_n-th root of unity ζ_n in Φ_n so that $\zeta_{n+1}^p=\zeta_n$, and put $\pi_n=1-\zeta_n$; π_n generates the unique prime ideal $\mathfrak{p}_n$ in the ring $\mathfrak{o}_n$ of local integers in Φ_n. Let $(\alpha, \beta)_n$ denote Hilbert's norm residue symbol in Φ_n for the power q_n and let

$$(\alpha, \beta)_n=\zeta_n^{[\alpha,\beta]_n}$$

with $[\alpha, \beta]_n$ in $\boldsymbol{Z}_p$, well determined mod q_n. The classical formulas for $[\alpha, \beta]_n$ state that

$$[\zeta_n, \beta]_n=q_n^{-1}T_n(\log\beta), \qquad \beta\in 1+\mathfrak{p}_n, \quad n\geqq 0,$$

$$[\pi_n, \beta]_n=-q_n^{-1}T_n(\zeta_n\pi_n^{-1}\log\beta), \qquad \beta\in 1+\mathfrak{p}_n, \quad n\geqq 0,$$

$$[\alpha, \beta]_0=-q_0^{-1}T_0(\zeta_0\alpha^{-1}\frac{d\alpha}{d\pi_0}\log\beta), \qquad \alpha\in 1+\mathfrak{p}_0, \quad \beta\in 1+\mathfrak{p}_0^2,$$

where T_n denotes the trace from Φ_n to $\boldsymbol{Q}_p$[1].

In a previous note [7], we have announced formulas for $[\alpha, \beta]_n$ which generalize the above formulas of Artin-Hasse. In the present paper, we shall prove those formulas and then discuss some related results.

As in the above, we retain most of the notations introduced in our earlier paper [6]. In particular, we denote by N_n the norm from Φ_n to $\boldsymbol{Q}_p$, and by $T_{n,m}$ and $N_{n,m}$ the trace and the norm, respectively, from Φ_m to Φ_n, $m\geqq n\geqq 0$; for an automorphism σ of the union Φ of all Φ_n, $n\geqq 0$, we denote by $\kappa(\sigma)$ the

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1) See [1], [2], [5]. For the general theory of the norm residue symbol needed in the following, see [2], Chap. 12 or [4], II, § 11, § 19. It is to be noted that the symbol $(\alpha, \beta)_n$ in [2] is the inverse of the same symbol in [4]. Here we follow the definition of $(\alpha, \beta)_n$ in [2] as we did in [7].