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On regularity of solutions of abstract differential equations of parabolic type in Banach space

By Hiroki TANABE

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The object of the present paper is to establish some estimates for the derivatives of the solution $u(t)$ of the abstract differential equation

$$du(t)/dt+A(t)u(t)=f(t) \tag{0.1}$$

of parabolic type in a Banach space X. Let $\{M_k\}$ be a sequence of positive numbers which has the properties specified later (cf. [2], [5], [6]; e.g. $M_k =(k!)^{\sigma}, \sigma\geqq 1$). Assuming that $A(t)$ belongs to the class $\{M_k\}$ ([5], [6]) as a function of t in some sense, we shall prove that $u(t)$ is also a function of t of the class $\{M_k\}$ provided that $f(t)$ is of the same class (see (0.3), (0.4) below). This sort of problem was investigated in [5] and [6] for wide classes of equations in Hilbert spaces including equations of parabolic type, of hyperbolic type, of Schrödinger type, etc.; all of these equations discussed there are associated with certain sesquilinear forms defined in some dense subspace. The authors of these papers investigate solutions belonging to the spaces $\mathfrak{D}_+(X)$, $\mathfrak{D}'_+(X)$, etc. ($\mathfrak{D}_+(X)$ and $\mathfrak{D}'_+(X)$ are the set of X-valued C^∞ functions and distributions respectively vanishing identically in $t<a$ for some $a\in(-\infty, \infty)$; consequently smooth solutions investigated there satisfy so many initial conditions $u^{(k)}(t_0)=0$, $k=1, 2, \cdots$, at a certain time t_0. For this reason quasi-analytic cases were naturally excluded so that the space $(\mathfrak{D}_{+,M_k}(X))$, the set of all functions of the class $\{M_k\}$ and belonging to $\mathfrak{D}_+(X)$, might contain sufficiently many elements. In the present paper only equations of parabolic type are concerned; however, the basic space X may be an arbitrary Banach space and furthermore quasi-analytic cases are equally treated since we investigate solutions satisfying (0.1) in the ordinary sense imposing upon them only the ordinary initial condition at an initial time.

The greater part of the paper is occupied by estimating the derivatives of the evolution operator $U(t, s)$ which is a bounded-operator-valued function satisfying

$$(\partial/\partial t)U(t, s)+A(t)U(t, s)=0\,, \qquad 0\leqq s<t\leqq T\,,$$
$$U(s, s)=I\,, \qquad 0\leqq s\leqq T\,,$$