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Nonlinear semigroups and evolution equations

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Introduction

This paper has been motivated by a recent paper by Y. Kōmura [3], in which a general theory of semigroups of nonlinear contraction operators in a Hilbert space is developed. Owing to the generality of the problem, Kōmura is led to consider multi-valued operators as the infinitesimal generators of such semigroups, which makes his theory appear somewhat complicated.

The object of the present paper is to restrict ourselves to single-valued operators in a Banach space X and to construct the semigroups generated by them in a more elementary fashion. Furthermore, we are able to treat, without essential modifications, time-dependent nonlinear equations of the form

$$du/dt+A(t)u=0, \qquad 0\leqq t\leqq T, \tag{E}$$

where the unknown $u(t)$ is an X-valued function and where $\{A(t)\}$ is a family of nonlinear operators with domains and ranges in X. In particular we shall prove existence and uniqueness of the solution of (E) for a given initial condition.

The basic assumptions we make for (E) are that the adjoint space X^* is uniformly convex and that the $A(t)$ are m-monotonic operators (see below), together with some smoothness condition for $A(t)$ as a function of t. We make no explicit assumptions on the continuity of the operators $A(t)$.

Here an operator A with domain $D(A)$ and range $R(A)$ in an arbitrary Banach space X is said to be *monotonic* if

$$\|u-v+\alpha(Au-Av)\|\geqq\|u-v\| \qquad \text{for every } u, v\in D(A) \text{ and } \alpha>0. \tag{M}$$

This implies that $(1+\alpha A)^{-1}$ exists and is Lipschitz continuous provided $\alpha>0$, where $1+\alpha A$ is the operator with domain $D(A)$ which sends u into $u+\alpha Au$. It can be shown (see Lemma 2.1) that $(1+\alpha A)^{-1}$ has domain X either for every $\alpha>0$ or for no $\alpha>0$; in the former case we say that A is *m-monotonic*.

The monotonicity thus defined can also be expressed in terms of the *duality*

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