Nonlinear semigroups and evolution equations

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Introduction

This paper has been motivated by a recent paper by Y. Kōmura [3], in which a general theory of semigroups of nonlinear contraction operators in a Hilbert space is developed. Owing to the generality of the problem, Kōmura is led to consider multi-valued operators as the infinitesimal generators of such semigroups, which makes his theory appear somewhat complicated.

The object of the present paper is to restrict ourselves to single-valued operators in a Banach space $X$ and to construct the semigroups generated by them in a more elementary fashion. Furthermore, we are able to treat, without essential modifications, time-dependent nonlinear equations of the form

$$(E) \quad \frac{du}{dt} + A(t)u = 0, \quad 0 \leq t \leq T,$$

where the unknown $u(t)$ is an $X$-valued function and where $\{A(t)\}$ is a family of nonlinear operators with domains and ranges in $X$. In particular we shall prove existence and uniqueness of the solution of (E) for a given initial condition.

The basic assumptions we make for (E) are that the adjoint space $X^*$ is uniformly convex and that the $A(t)$ are m-monotonic operators (see below), together with some smoothness condition for $A(t)$ as a function of $t$. We make no explicit assumptions on the continuity of the operators $A(t)$.

Here an operator $A$ with domain $D(A)$ and range $R(A)$ in an arbitrary Banach space $X$ is said to be monotonic if

$$(M) \quad \|u-v+\alpha(Au-Av)\| \geq \|u-v\| \quad \text{for every } u, v \in D(A) \text{ and } \alpha > 0.$$

This implies that $(1+\alpha A)^{-1}$ exists and is Lipschitz continuous provided $\alpha > 0$, where $1+\alpha A$ is the operator with domain $D(A)$ which sends $u$ into $u+\alpha Au$. It can be shown (see Lemma 2.1) that $(1+\alpha A)^{-1}$ has domain $X$ either for every $\alpha > 0$ or for no $\alpha > 0$; in the former case we say that $A$ is m-monotonic.

The monotonicity thus defined can also be expressed in terms of the duality

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