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On $\aleph_0$-complete cardinals

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In [4], D. Scott proved that, if we assume $V=L$ and the existence of a measurable cardinal number in the set theory Σ^* of [1], then we have a contradiction.

The main purpose of this parer is to investigate on the problem concerning to certain kind of constructibility and the existence of $\aleph_0$-complete cardinal numbers (2-valued measurable cardinal numbers). In view of this point, we first remark that if the system Σ^*, $\exists xT(x)$ is consistent, then the system

$$\Sigma^*, \exists y(T(y) \wedge \exists x(V=L_x \wedge Od_x``x \subset 2^y))$$

is consistent, where $T(y)$ is the statement that there is a non-principal $\aleph_0$-complete ultrafilter over the set y whose character is cardinal number y, and L_x is the class constructed from the set x by Lévy's method in [2].

In this paper we prove the following several results:

1) The system $\Sigma^*, \exists y(T(y) \wedge \exists x(V=L_x \wedge Od_x``x \subset y))$ is not consistent.

2) Let $\Phi(a)$ be a standard defining postulate defined later. Then the system $\Sigma^*, \exists x(T(x) \wedge \Phi(x))$ is not consistent.

Remark that, as is well known, all of the defining postulates of the following cardinals are standerd: $\aleph_0, \aleph_1, \cdots, \aleph_\omega, \cdots$; the first one of weakly inaccessible cardinal, strongly inaccessible cardinal, hyper-inaccessible cardinal; the first cardinal α such that α is hyper-inaccessible of type α; and so on.

Concerning to this kind of results, I would like to propose the following problem: For what kind of formula $A(a)$, is the system $\Sigma^*, \exists x(T(x) \wedge A(a))$ not consistent? Especially what will happen for the formulas $\exists x(V=L_x \wedge \sup(Od_x``x) < 2^{\bar{\bar{a}}})$ or $\exists x(V=L_x \wedge \sup(Od_x``x) < a^+)$ where a^+ is the smallest cardinal number strictly greater than a.

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1. We shall begin by introducing several notations and the terminology.

DEFINITION. An ultrafilter $\mathcal{F}$ is said to be $\aleph_\alpha$-complete, if the following condition is satisfied:

$$\text{if } A_\nu \in \mathcal{F} \text{ for each } \nu \in I, \text{ then } \bigcap_{\nu \in I} A_\nu \in \mathcal{F}, \text{ where } \bar{\bar{I}} \leqq \aleph_\alpha .$$