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Harmonic forms and Betti numbers of certain contact Riemannian manifolds

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Let M be a compact regular contact manifold, then M can be considered as a prinicipal circle bundle over the symplectic manifold $B=M/\xi$. And we can define a Riemannian metric in M so that M is a K-contact Riemannian manifold. More special K-contact Riemannian manifold is a normal contact Riemannian manifold or Sasakian manifold. With respect to this Riemannian metric we study the properties of harmonic forms on M. When M is a compact regular K-contact Riemannian manifold the fibering $M\rightarrow B$ gives the standard model to study the Betti numbers and harmonic forms. In particular if M is a compact regular normal contact Riemannian manifold, $M\rightarrow B$ is the most typical, since B is kählerian and a Hodge manifold. By these models, we get some results on harmonic forms and Betti numbers of K-contact Riemannian manifolds which are not necessarily regular. In § 1, we give the fundamental relations satisfied by the structure tensors, and next the relations of the Betti numbers. In § 3 and § 4, the relations of harmonic forms on M and B are given. In § 5, we devide K-contact Riemannian manifolds into two classes according as M is an η-Einstein space or not, and study respective cases. Manifolds are assumed to be of class C^{∞} and connected.

§ 1. Preliminary.

We denote by ϕ, ξ, η, w and g the structure tensors of an m-dimensional ($m\geqq 3$) contact Riemannian manifold M, where ϕ, ξ, η and w are tensor fields on M of type (1,1), (1,0), (0,1) and (0,2) respectively and g is the metric tensor. They satisfy the following relations:

(1.1) $$\phi\xi=0\,,\qquad \eta(\xi)=1\,,\qquad \phi^{2}u=-u+\eta(u)\xi\,,$$

(1.2) $$\eta(u)=g(u,\xi)\,,\qquad g(\phi u,\phi v)=g(u,v)-\eta(u)\eta(v)\,,$$

(1.3) $$d\eta(u,v)=2g(u,\phi v)=2w(u,v)$$

for any vector fields u and v on M. If we denote by ∇ and δ the Riemannian

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