J. Math. Soc. Japan
Vol. 19, No. 2, 1967

On generalized graded Lie algebras and geometric structures I

By Noboru TANAKA

(Received Nov. 8, 1966)

Introduction.

The main purpose of the present paper is to establish a new prolongation theorem for certain linear group structures which are of infinite type and even not elliptic, and is to apply this theorem to the geometry of differential systems (distributions in the sense of Chevalley) and the geometry of real submanifolds in complex manifolds. (A linear group G is called elliptic if the Lie algebra $\mathfrak{g}$ of G contains no matrix of rank 1, and a G-structure is called elliptic if the linear group G is elliptic. A recent work of Ochiai [4] has shown that a G-structure is elliptic if and only if the defining equation of infinitesimal automorphisms of the G-structure forms an elliptic system of linear differential equations).

First of all, we introduce the notion of a generalized graded Lie algebra (Def. 2.1). Let $\mathfrak{g}$ be a Lie algebra, and let $(\mathfrak{g}_p)_{p\in\boldsymbol{Z}}$ be a family of subspaces of $\mathfrak{g}$, $\boldsymbol{Z}$ being the additive group of integers, which satisfies the following conditions:

1) $\mathfrak{g}=\sum_p \mathfrak{g}_p$ (direct sum);
2) $\dim \mathfrak{g}_p<\infty$;
3) $[\mathfrak{g}_p, \mathfrak{g}_q]\subset \mathfrak{g}_{p+q}$.

Under these conditions, we say that the direct sum $\mathfrak{g}=\sum_p \mathfrak{g}_p$ is a generalized graded Lie algebra or simply a graded Lie algebra. (Note that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and the mappings $\mathfrak{g}_0\times\mathfrak{g}_p\ni(X_0, X_p)\rightarrow[X_0, X_p]\in\mathfrak{g}_p$ define representations ρ_p of the Lie algebra $\mathfrak{g}_0$ on the vector spaces $\mathfrak{g}_p$.)

Now consider a graded Lie algebra of the form $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$, where we must think $\mathfrak{g}_p$ ($p<-2$ or $p>0$) of vanishing, and assume the following conditions:

1° $\mathfrak{g}_{-2}=[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$;

2° the representation ρ_{-1} of the Lie algebra $\mathfrak{g}_0$ on the vector space $\mathfrak{g}_{-1}$ is faithful.

It is shown that there corresponds to such a graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ a