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Isometric immersions of Riemannian manifolds

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Introduction.

It seems to be natural in differential geometry to conjecture that many theorems proved under the condition that the underlying manifolds are compact can be extended to manifolds with complete Riemannian metric under some suitable additional conditions.

Now, it is clear that for any smooth function f on a compact manifold there is a point p such that $\nabla_i f(p)=0$ and $\nabla_i\nabla_j f(p)$ is negative semi-definite. It is also clear that if a smooth function $f(x)$ on a real line R has an upper bound, then for any $\varepsilon>0$ there is $x\in R$ such that $f'(x)$ and $f''(x)$ are $<\varepsilon$. This simple fact, however, can not be extended in general to a complete Riemannian manifold. That is, there is a complete Riemannian manifold M and a bounded function f on M such that $m(p)=\{X^iX^j\nabla_i\nabla_j f(p)\,;\,\|X\|=1\}$ is always larger than $a>0$. This example can easily be constructed on R^2 with metric $dr^2+g(r)d\theta^2$ (in the polar coordinate expression). Let $f(r,\theta)=f(r)=\frac{r^2}{1+r^2}$. Since

$$\nabla\nabla f(=\nabla_i\nabla_j f dx^i dx^j)=f''(r)dr^2+\frac{1}{2}f'(r)g'(r)d\theta^2\,,$$

one can choose a suitable function $g(r)$ so that it satisfies (a) $g(r)$ is smooth and $g(r)=r$ for $0\leqq r<1/2$, (b) $g(r)$ is a solution of $g'(r)/g(r)=2c/f'(r)$ (for example $g(r)=\exp\int_1^r c(1-r)^2/r\,dr)$ for $r\geqq 1$. In this example, one can see easily that the sectional curvature has no lower bound.

In this paper, there will be proved first of all a generalization of this example, that is:

THEOREM A. *Let M be a connected and complete Riemannian manifold whose sectional curvature $K(X,Y)$ has a lower bound i.e. $K(X,Y)\geqq -K_0$. If a smooth function f on M has an upper bound, then for any $\varepsilon>0$, there is a point $p\in M$ such that $\|\operatorname{grad} f(p)\|<\varepsilon$ and $m(p)=\max\{X^iX^j\nabla_i\nabla_j f(p)\,;\,\|X\|=1\}$ $<\varepsilon$.*

For an application of this theorem, an isometric immersion of M into the Euclidean N-space R^N will be considered. It is clear that if M is compact,