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On the relative class number of finite algebraic number fields

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Let l be an odd prime number. The relative class number, the so-called first factor h_n^- of the class number of the cyclotomic field generated by a primitive l^{n+1}-th root of unity over the rational number field is given by the well-known formula ($n \geqq 0$):

$$h_n^- = 2l^{n+1} \prod_{\chi} \left(-\frac{1}{2l^{n+1}} \sum_{m} m\chi^{-1}(m) \right),$$

where m ranges over all integers satisfying $0 \leqq m < l^{n+1}$, $(m, l) = 1$, and χ over all characters of the multiplicative group of integers mod l^{n+1} with $\chi(-1) = -1$[1]. According to this formula, it can be observed that h_n^- is divisible by h_0^-. Let L and M be totally imaginary quadratic fields over a totally real algebraic number field L_0 and M_0, respectively. Let further L and L_0 be subfields of M and M_0 respectively. Can it be proved further that the relative class number of M/M_0, i.e. the ratio of the class number of M to that of M_0 is divisible by the relative class number of L/L_0 in such a case? (Both relative class numbers of M/M_0 and L/L_0 are rational integers (cf. Chevalley [**2**]).) The main purpose of this paper is to consider this problem in more general cases. The main results are as follows. Let E and F be finite extensions of a finite algebraic number field k such that E is a Galois extension of k and $E \cap F = k$. We shall show that if there exists no non-trivial unramified abelian extension of F contained in the composite field EF, then for any prime number p prime to the relative degree of F/k, the p-part of the relative class number of F/k is less than or equal to the p-part of the relative class number of EF/E (Theorem 1). (In this paper, "an unramified abelian extension of F" means a subfield of the Hilbert's class field over F.) As an interesting consequence of this, we shall show that for any totally real algebraic number field L_0 of finite degree and any rational integer n prime to the degree of L_0, there are infinitely many totally imaginary quadratic extensions L of L_0 so that the relative class

1) See Iwasawa [**5**], in which the class number formula is used in this formula: the formula in Hasse [**3**] is slightly different from this formula.