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On the exponential decay of solutions for some partial differential equations

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In [1] C. Morawetz has shown, using the energy integrals (the a-b-c method initiated by K. Friedrichs), that a solution of the wave equation

$$u_{tt}-\Delta u+\alpha u=0\,, \qquad \alpha>0\,,$$

which vanishes in the forward light cone $|x|<t$ $(t>0)$ and has finite energy

$$\int_{t=0}(|\nabla u|^2+u_t^2+\alpha u^2)dv<\infty\,,$$

vanishes identically.

The purpose of the present note is to show that the above result can be extended in the following sharpened form to the case of partial differential equations of more general type: Let u be a solution for the evolution equations of certain types, e. g. the Klein-Gordon wave equation or the Schrödinger wave equation. If this solution u decays exponentially with time on any compact set, then the solution u must vanish identically. The author wishes to express his hearty thanks to Professor K. Yosida who kindly pointed out mistakes in the manuscript and gave him many valuable advices, and also to Professor H. Fujita for his many valuable suggestions.

§ 1. Notations and results.

We denote by G a whole space E^n or the exteriors (or interiors) of bounded $(n-1)$-dimensional hypersurfaces which are locally of class C^4. Let $x=(x_1, \cdots, x_n)$ be the generic point of E^n whose length $\sqrt{x_1^2+\cdots+x_n^2}$ is denoted by $|x|$, $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ be a general derivative where $D_j=\dfrac{\partial}{\partial x_j}$ and α stands for the multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ whose length $\alpha_1+\cdots+\alpha_n$ is also denoted by $|\alpha|$. Let $\|\cdot\|$ be the norm of $L^2(G)$, $(\cdot, \cdot)$ be the scalar product in $L^2(G)$ and $\|\cdot\|_{L^2(K)}$ be the norm of $L^2(K)$. Let H_j be the totality of those complex-valued functions v in $L^2(G)$ for which the distribution derivatives $D^\alpha v$ also lie in $L^2(G)$ for $|\alpha|\leqq j$. Then H_j is a Hilbert space under the norm

$$\|v\|_j=\{\sum_{|\alpha|\leqq j}\|D^\alpha v\|^2\}^{\frac{1}{2}}\,.$$