J. Math. Soc. Japan
Vol. 19, No. 1, 1967

A problem on the existence of an Einstein metric

By Tadashi NAGANO

(Received June 21, 1966)

This note is to present a variational problem, to which the problem of the existence of an Einstein metric is reduced. The latter has called certain geometers' attention from various points of view; for instance, if an Einstein metric is proved to exist on a compact simply connected 3-manifold then the Poincaré conjecture for the 3-sphere is answered affirmatively (see e. g. [**1**]). The variational problem arises from the characterization of Einstein metrics in the following theorem.

THEOREM. *Let M be an n-dimensional compact orientable manifold with a fixed volume element Ω (or an $SL(n, \mathbf{R})$-structure P). And let $\mathfrak{G}$ be the family of all Riemannian metrics with Ω on M (or all $SO(n)$-structures contained in P). Then a Riemannian metric g_0 in $\mathfrak{G}$ is Einstein if and only if the function $I = I(g) = \int_M R\Omega$ on $\mathfrak{G}$ attains a critical value at g_0, where R is the scalar curvature of g.*

Before the proof some comments will be adequate. An $SL(n, \mathbf{R})$-structure is essentially unique on M [**2**]. I attains its critical value at g_0 by definition if one has $D(I(g(t))) = g_0$ at $t = 0$, $D = d/dt$, for any differentiable one-parameter family $\{g(t)\}$ with $g(0) = g_0$ of Riemannian metrics in $\mathfrak{G}$. The theorem is trivial when $n = 2$, since I is then a constant by the Gauss-Bonnet formula.

Now the proof is given by a straightforward tensor calculus. We put $a = Dg(t)$. Such an $a = (a_{ij})$ is characterized as a symmetric covariant tensor field of degree 2 whose "trace" $a_a^a = g^{ij}a_{ij}$ vanishes identically due to the assumption on the volume element. It is easy to obtain $DR_{ij} = \nabla_a D\left\{ {a \atop i\ j} \right\}$ by noting $D\left\{ {a \atop a\ j} \right\} = 0$, so that we have $DI(g) = D\int_M R\Omega = \int_M (DR)\Omega = \int_M (g^{ab}DR_{ab} + R_{ab}Dg^{ab})\Omega = \int_M \left(\nabla_a g^{bc} D\left\{ {a \atop b\ c} \right\} - R_{ab}g^{ab}\right)\Omega = -\int_M R_{ab}a^{ab}\Omega$, since the second integrand in the third integral is the divergence of the vector field $(v^i) = \left(g^{bc}D\left\{ {i \atop b\ c} \right\}\right)$. Thus $DI(g)$ vanishes if and only if $R_{ab}a^{ab}$ vanishes everywhere on M. This occurs when and only when (R_{ab}) is a scalar multiple of g. The theorem is proved.