J. Math. Soc. Japan
Vol. 18, No. 4, 1966

# Linear differential systems with singularities and an application to transitive Lie algebras

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(Received July 1, 1966)

## Introduction

In this note we will first define linear differential systems *with* singularities, and then give a sufficient condition for their integrability (Theorem 1 below), similar to Frobenius' theorem. As an application we will prove that a compact manifold $M$ is completely determined by the Lie algebra $L(M)$ of all analytic vector fields on $M$ along with an isotropy subalgebra of $L(M)$ (Theorem 2). On this regard we recall that Myers [4] proved that $M$ is completely determined by the ring $F(M)$ of all analytic functions on $M$.

We shall always be in the analytic category. The Lie algebra $L(M)$ of the vector fields is a module over the function ring $F(M)$.

DEFINITION. *A linear differential system* on $M$ is an $F(M)$-submodule of $L(M)$.

This notion is a generalization of the usual linear differential system (without singularities), i.e., a vector subbundle of the tangent bundle $T(M)$ (indeed any real analytic vector bundle has sufficient ample sections, [5]), or a "distribution" as Chevalley calls [2]. A vector subspace $L$ of $L(M)$ over the reals $\boldsymbol{R}$ generates a unique $F(M)$-module $L^F$. We shall understand $L^F$ under the linear differential system of $L$. Given a subspace $L$ of $L(M)$ and a point $x$ of $M$, we put $L(x)=\{u(x) \mid u \in L\}$ and $L_x=\{u \in L \mid u(x)=0\}$ $(0 \rightarrow L_x \rightarrow L \rightarrow L(x) \rightarrow 0$ is exact). $L(x)$ is a subspace of the tangent space $T_x(M)$ to $M$ at $x$, which we call *the integral element* of $L$ at $x$. *An integral manifold* $N$ of $L$ is by definition a connected submanifold of $M$ such that each tangent space to $N$ is an integral element of $L$; $T_x(N)=L(x)$, $x \in N$. The linear differential system $L^F$ of $L$ has the same integral elements and integral manifolds as $L$. If $L$ is a Lie subalgebra of $L(M)$, then so is $L^F$ and $L_x$ is a subalgebra of $L$, which we call *the isotropy subalgebra* of $L$ at $x$.

THEOREM 1. *If $L$ is a Lie subalgebra of $L(M)$, then $M$ has a unique partition $\mathfrak{N}=\{N\}$ by integral manifolds $N$ of $L$.* (That is, $M$ is the disjoint union

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* A part of this paper was prepared when the author was at University of California, Berkeley, partially supported by NSF GP-2497.