J. Math. Soc. Japan
Vol. 18, No. 4, 1966

Solvable groups with isomorphic group algebras

By Tadao OBAYASHI

(Received April 13, 1966)

1. Several authors have studied the interesting problem whether non-isomorphic groups can have isomorphic integral group algebras. J. A. Cohn and D. Livingstone [1], D. B. Coleman [2] and D. S. Passman [3] gave partial answers in the case of nilpotent groups. Their results seem to be based mainly on the fact that the center of a group is determined by its group algebra.

In this paper we intend to show that the derived groups of a group are determined by its group algebra. Thus for solvable groups we can prove

MAIN THEOREM. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be finite groups with isomorphic group algebras over the ring of all integers in a finite algebraic number field. If $\mathfrak{G}$ is solvable, then $\mathfrak{H}$ is solvable with the same length of the derived series as that of $\mathfrak{G}$ and the factor groups of their derived series are isomorphic.*

Throughout this paper R denotes the ring of all integers in a finite algebraic number field and Z the ring of rational integers. For an arbitray finite group $\mathfrak{G}$ the group algebra $R(\mathfrak{G})$ (resp. $Z(\mathfrak{G})$) is the algebra over R (resp. Z) with a free basis multiplicatively isomorphic with $\mathfrak{G}$. We shall often identify the elements of this basis with the elements of $\mathfrak{G}$.

2. Let $\mathfrak{G}$ be a finite group. Then the group algebra $R(\mathfrak{G})$ is an augmented algebra with the unit augmentation $\eta_{\mathfrak{G}}$ and the augmentation ideal $I(\mathfrak{G})$ is a two-sided ideal in $R(\mathfrak{G})$ with R-free basis $g-1$, $g\neq 1$, $g\in\mathfrak{G}$, where 1 denotes the identity element of $\mathfrak{G}$.

LEMMA 1. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be finite groups. If φ; $R(\mathfrak{G})\cong R(\mathfrak{H})$ is an isomorphism as algebras, then there exists a group $\mathfrak{G}'$ consisting of unit elements of finite order in $R(\mathfrak{G})$ such that $\mathfrak{G}'\cong\mathfrak{G}$, $R(\mathfrak{G}')=R(\mathfrak{G})$ and $\eta_{\mathfrak{G}'}=\eta_{\mathfrak{H}}\circ\varphi$.*

PROOF. For each $g\in\mathfrak{G}$, $\eta_{\mathfrak{H}}\circ\varphi(g)$ is a unit of finite order in R. Then we see easily that $\mathfrak{G}'=\{g'=(\eta_{\mathfrak{H}}\circ\varphi(g))^{-1}g\in R(\mathfrak{G})\,;\,g\in\mathfrak{G}\}$ is the desired group.

In view of this lemma, we shall always assume implicitly that an isomorphism φ; $R(\mathfrak{G})\cong R(\mathfrak{H})$ of group algebras is compatible with augmentation maps; i. e., $\eta_{\mathfrak{G}}=\eta_{\mathfrak{H}}\circ\varphi$.

PROPOSITION 1. *Any algebra isomorphism φ; $R(\mathfrak{G})\cong R(\mathfrak{H})$ induces the ring isomorphism $I(\mathfrak{G})\cong I(\mathfrak{H})$.*