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On semisimple extensions and separable extensions over non commutative rings

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Introduction

In this paper we intend to generalize the semisimplicity and separability for algebras to the case where the basic ring is non commutative. Separable algebras in the sense of Auslander-Goldman [2] and semisimple algebras in the sense of Hattori [5] are separable extensions and semisimple extensions respectively in our sense. For the definitions of these the relative homological algebra introduced by Hochshild [8] is useful.

In § 1 we shall define the *semisimple extension* over non commutative rings and study about some properties of them. In § 2 we study about *separable extensions*. Separable extensions are semisimple extensions (Proposition 2.6). If Λ is an algebra over a commutative ring R and Γ is a subalgebra, then Λ is a separable extension of Γ if, and only if, $\Lambda\otimes_R\Delta$ is a semisimple extension of $\Gamma\otimes_R\Delta$ for every R-algebra Δ (Corollary 2.16). In § 3 some examples are given. The Galois extension of a non commutative ring, in the sense of T. Kanzaki, is a separable extension (Proposition 3.3).

Throughout this paper we assume that all rings have the identity, all subrings contain this element and all modules are unitary.

§ 1. Semisimple extension

The basic notions of relative homological algebra are introduced by G. Hochschild [8]. We recall these briefly.

Let Λ be a ring, Γ a subring of Λ. A sequence of (left) Λ-modules is (Λ, Γ)-exact if it is exact and splits as a sequence of Γ-modules. Consider a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & P & & \end{array}$$

where the row is (Λ, Γ)-exact. P is (Λ, Γ)-projective if there exists a Λ-homomorphism of P into M such that the diagram is commutative. For any