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An exponential formula for one-parameter semi-groups of nonlinear transformations

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For a complete normed linear space S consider a function T from $[0, \infty)$ to the set of continuous transformations from S to S which satisfies:

(1) $T(x)T(y)=T(x+y)$ if $x, y>0$,

(2) $\| T(x)p-T(x)q \| \leqq \| p-q \|$ if $x \geqq 0$, p, q are in S,

(3) if p is in S and $g_p(x)=T(x)p$ for all x in $[0, \infty)$ then g_p is continuous and $\lim_{x\to 0+} g_p(x)=p$.

If it is also specified that $T(x)$ is linear for all $x \geqq 0$, then one has a semi-group about which the following is known ([1] chapters 10, 11 and [3] sections 142, 143):

For all p in some dense subset of S, $g'_p(0)$ exists and if $Ap=g'_p(0)$ for all p for which $g'_p(0)$ exists, then $(I-xA)^{-1}$ exists, has domain S and is continuous for all $x \geqq 0$. Moreover, if p is in S and $x \geqq 0$,

$$(*) \qquad \lim_{n\to\infty} \| (I-(x/n)A)^{-n}p-T(x)p \| =0 .$$

It is the purpose of this note to add to assumptions (1)-(3) a differentiability condition (which, it turns out, holds in the linear special case) which implies an "exponential formula" suggested by (*). The results of this note give a nonlinear version of the linear strong case of [1] (section 11.5); previous work [2] (section 3) of this author gave a nonlinear version of the linear uniform case of [1] (section 11.2).

The differentiability condition mentioned above is:

(4) there is a dense subset D of S such that if p is in D, then g'_p is continuous with domain $[0, \infty)$.

If $\delta>0$, denote $(1/\delta)[T(\delta)-I]$ by A_δ. The main result of this note follows.

THEOREM. *If* (1)-(4) *hold, p is in S and $x \geqq 0$, then*

$$\lim_{n\to\infty} \limsup_{\delta\to 0+} \| (I-(x/n)A_\delta)^{-n}p-T(x)p \| =0 .$$

Consider first some lemmas.

LEMMA 1. *Under conditions* (1)-(3), *if $\delta, x>0$, then $(I-xA_\delta)^{-1}$ exists and has domain S.*