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Homomorphic images of Lie groups

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§ 0. Introduction.

In their suggestive paper [3], Gleason and Palais studied some fundamental properties of homeomorphism group $H(M)$ of a manifold M and then proposed a problem: *Does the closure of a homomorphic image of any connected Lie group in* $H(M)$ *necessarily become a Lie group?* The topology for $H(M)$ is of course the compact open topology, which is known to give an important example of non locally compact group. This problem, however, seems to be still far from the final answer.

Another open problem related to this is the one, raised by Montgomery and Zippin [6], that states: *Does a locally compact subgroup of* $H(M)$ *necessarily become a Lie group?*

Being suggested by these two problems, we are led to consider the following similar to but weaker than that of Gleason-Palais':

(1) *Is the closure of homomorphic image of any connected Lie group in* $H(M)$ *necessarily locally compact?*

Our main concern in this paper is to investigate this problem from several points of view and solve it in the case of one dimensional manifolds.

First we are interested in knowing to what extent the problem (1) reflects the characteristic properties of the homeomorphism group $H(M)$ instead of general topological groups, which primarily implies the following:

(2) *Are there a topological group* H *and a connected Lie group* G *such that the closure of a homomorphic image of* G *in* H *is not locally compact?*

An answer to this problem is seen in [7], namely we have shown the existence of such a topological group by giving an extraordinary topology to a real line.

Now the problem (1) naturally has two versions: (a) one is to characterize the class $\mathfrak{A}$ of Lie groups whose monomorphic images in $H(M)$ have locally compact closures, and (b) the other is to characterize the class $\mathfrak{B}$ of topological groups in which any homomorphic images of any connected Lie groups have the locally compact closures.

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