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Computation of invariants in the theory of cyclotomic fields

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1. Let a prime number p be fixed, and let F_n, $n \geqq 0$, denote the cyclotomic field of p^{n+1}-th roots of unity over the rational field $\boldsymbol{Q}$. Let $p^{c(n)}$ be the highest power of p dividing the class number h_n of F_n. Then there exist integers λ_p, μ_p, and ν_p (λ_p, $\mu_p \geqq 0$), depending only upon p, such that

$$c(n) = \lambda_p n + \mu_p p^n + \nu_p ,$$

for every sufficiently large integer n[1]. In the present paper, we shall determine, by the help of a computer, the coefficients λ_p, μ_p, and ν_p in the above formula for all prime numbers $p \leqq 4001$. We shall see in particular that $\mu_p = 0$ for every $p \leqq 4001$. Let S_n denote the Sylow p-subgroup of the ideal class group of F_n. For the above primes, we shall determine not only the order $p^{c(n)}$ of S_n but also the structure of the abelian group S_n for every $n \geqq 0$.

Let $p = 2$. Then we know by Weber's theorem that $c(n) = 0$, $S_n = 1$ for any $n \geqq 0$ so that $\lambda_2 = \mu_2 = \nu_2 = 0$. Therefore, we shall assume throughout the following that p is an odd prime, $p > 2$.

2. Let $\boldsymbol{Q}_p$ and $\boldsymbol{Z}_p$ denote the field of p-adic numbers and the ring of p-adic integers, respectively. Let F be the union of all fields F_n, $n \geqq 0$. Then F is an abelian extension of $\boldsymbol{Q}$, and we denote the Galois group of $F/\boldsymbol{Q}$ by G. For each p-adic unit u in $\boldsymbol{Q}_p$, there is a unique automorphism σ_u of F such that $\sigma_u(\zeta) = \zeta^u$ for any root of unity ζ in F with order a power of p. The mapping $u \to \sigma_u$ then defines a topological isomorphism of the group of p-adic units in $\boldsymbol{Q}_p$ onto the compact abelian group G. Let Γ and Δ denote the subgroups of G corresponding to the group of 1-units in $\boldsymbol{Q}_p$ and the group $\boldsymbol{V}$ of all ($p-1$)-st roots of unity in $\boldsymbol{Q}_p$, respectively. Then we have

$$G = \Gamma \times \Delta ;$$

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1) For the results on cyclotomic fields used in the present paper, see K. Iwasawa, On the theory of cyclotomic fields, Ann. of Math., **70** (1959), 530-561; K. Iwasawa, On some modules in the theory of cyclotomic fields, J. Math. Soc. Japan, **16** (1964), 42-82.