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Transformation groups satisfying some local metric conditions

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If a locally compact group G acts effectively on a manifold M, then is G necessarily a Lie group? Considerably many investigations have been directed to this well known problem. The purpose of this paper is to show that it is affirmatively answered under some local metric conditions: locally Lipschitzian (cf. Definition 1) or locally similar (cf. Definition 3).

The proof is reduced to show that if G is zero-dimensional compact then for an open normal subgroup G' of G there exists a G'-invariant local metric ρ^* in M such that $\rho^*(x, y) \leqq c \cdot \rho(x, y)$ holds locally, where c is a constant and ρ is a local euclidean metric.

In this paper a *manifold* means a separable, metric, connected, and locally euclidean space.

§1. Locally Lipschitzian transformation groups.

DEFINITION 1. A topological transformation group G acting on a manifold M is said to be *locally Lipschitzian* if for a coordinate neighborhood U_a of each point a in M there exist a neighborhood V of the identity of G and a neighborhood U of the point a as follows: 1) $V(U) \subset U_a$, 2) $\rho(g(x), g(y)) \leqq c \cdot \rho(x, y)$ for all $g \in V$ and all $x, y \in U$, where ρ is a euclidean distance function in U_a and c is a constant.

The following Lemma 1 shows that classical transformation groups acting on manifolds are locally Lipschitzian.

LEMMA 1. *In the above if G is locally compact and, in place of the condition 2), 2)′ the local coordinate functions of $g(x)$ have partial derivatives with respect to x that are continuous at $(g; x)$ simultaneously, then G is locally Lipschitzian.*

PROOF. Let n be the dimension of M. Choose a compact neighborhood V of the identity of G and a neighborhood U' of the point a that satisfy the conditions 1) and 2)′, and an open convex neighborhood U of a such that $\bar{U}$ is compact and $\bar{U} \subset U'$. The coordinate functions $g_i(x)$, $i=1, 2, \cdots, n$, are totally differentiable with respect to $x \in \bar{U}$ i.e., for any point $x+h$ (vector sum) sufficiently near x,