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Hardy-Littlewood majorants in function spaces

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1. Throughout this paper, we term $\boldsymbol{X}$ a *Banach function space*[1], if $\boldsymbol{X}$ is a *normed linear space of integrable functions over the interval* (0, 1) *satisfying*

(i) $|g| \leqq |f|$[2], $f \in \boldsymbol{X}$ *implies* $g \in \boldsymbol{X}$ *and* $\|g\| \leqq \|f\|$;

(ii) $0 \leqq f_n \uparrow_{n=1}^{\infty} f$ *implies* $\sup_{n \geqq 1} \|f_n\| = \|f\|$;

(iii) $0 \leqq f_n \uparrow_{n=1}^{\infty}$ *with* $\sup_{n \geqq 1} \|f_n\| < +\infty$ *implies* $\bigcup_{n=1}^{\infty} f_n \in \boldsymbol{X}$[3].

We shall call the rorm fulfiling (i) and (ii) to be *semi-continuous*. $\boldsymbol{X}$ is said to have the *Rearrangement Invariant property*[4] (or shortly *RIP*), if each function g equimeasurable to a function $f \in \boldsymbol{X}$ also belongs to $\boldsymbol{X}$ and $\|g\| = \|f\|$.

Let f be an integrable function on (0, 1). The *Hardy-Littlewood majorant* $\theta(f)$ of f is the function defined by

$$\theta(f)(x) = \sup_{0 \leqq y \leqq 1} \int_x^y \frac{f(t)}{y-x} dt \quad (x \in (0, 1)), \tag{1}$$

provided it exists almost everywhere. G. H. Hardy and J. E. Littlewood have shown that if $f \in \boldsymbol{L}^p (1 < p)$, then $\theta(f)$ is defined and belongs to $\boldsymbol{L}^p$ also [**9**]. Here, in accordance with G. Lorentz [**3**], we shall say that $\boldsymbol{X}$ has the *Hardy-Littlewood property*, and shall denote by $\boldsymbol{X} \in HLP$, if $f \in \boldsymbol{X}$ implies $\theta(f) \in \boldsymbol{X}$. In his paper cited above, G. Lorentz discussed this property for Banach function spaces having *RIP*[5], and presented necessary and sufficient conditions in order that $\boldsymbol{X} \in HLP$, in case $\boldsymbol{X}$ is an Orlicz space $\boldsymbol{L}_{\Phi}$ or a space $\Lambda(\phi)$.

The aim of this note is to give a necessary and sufficient condition in order that a general Banach function space $\boldsymbol{X}$ with *RIP* have the Hardy-Little-

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1) Here we deal with Banach spaces consisting of real functions. For an exposition of Banach function spaces see [**4**].

2) $|g| \leqq |f|$ means that $g(t) \leqq f(t)$ holds almost everywhere in (0, 1).

3) A norm satisfying (iii) is called monotone complete. If a norm is monotone complete, it is complete.

4) On account of Theorem 3 in [**8**], we may replace this condition by the weak rearrangement invariant property (this requires only $g \in X$, if g is equimeasurable to an $f \in X$) throughout this paper.

5) In his paper Banach function spaces are introduced in terms of Köthe spaces.