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On injective modules and flat modules

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§ 0. Introduction.

In this paper we will first establish a duality between injectivity and flatness for modules. That is: Let E be a faithfully injective module[1]. Then, for each module A, we have $W.\dim. A = Id\, \mathrm{Hom}(A, E)$ and further A is faithfully flat if and only if $\mathrm{Hom}(A, E)$ is faithfully injective. (Theorem 1.4). Moreover, when the ring is noetherian, we have $Id\, A = W.\dim \mathrm{Hom}(A, E)$ and A is faithfully injective if and only if $\mathrm{Hom}(A, E)$ is faithfully flat (Theorem 1.5). In this case we have also $Id\, A = Id\, A \otimes M$, where M is a faithfully flat module, and A is faithfully injective if and only if so is $A \otimes M$ (Theorem 1.3). In case of (semi-) local rings, from these results, we can prove that the self-injective dimension of the ring is equal to the weak dimension of the canonical injective module[1] of the ring.

Next, using above results, we will treat a problem: Is the tensor product of injective modules over a commutative ring again injective[2]? When the ring is an integral domain, we can easily say "Yes" (cf. [4]). If the ring is noetherian and every principal ideal is projective (or flat), the ring is a direct sum of a finite number of integral domains [**5**, Lemma 3], so in this case the answer is also affirmative. In case of noetherian rings, we will give a condition for an affirmative answer and its several equivalent conditions (Theorem 2.4).

§ 1. Duality.

Let R, S be two rings. We consider the situation $({}_RA, {}_RB_S, {}_SC)$, that is, A is a left R-module, B is a left R- right S- bimodule and C is a left S-module. Then we can define a homomorphism:

$$\tau : \mathrm{Hom}_R(A, B) \otimes_S C \to \mathrm{Hom}_R(A, B \otimes_S C)$$

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1) See [**6**] for the definition.

2) This problem was first proposed by N. Yoneda.