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A decomposition of Markov processes

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§ 1. Introduction.

Let S be a locally compact metric space and D be an open set having closure S and non-empty compact boundary $\partial D = S - D$. Let $\boldsymbol{B}(S)$ [$\boldsymbol{B}(\partial D)$] be the Borel field generated by all the closed sets in S [∂D], and $B(S)$ [$B(\partial D)$] be the space of real-valued bounded $\boldsymbol{B}(S)$-measurable [$\boldsymbol{B}(\partial D)$-measurable] functions on S [∂D]. Suppose that we are given a Markov process $X = (x_t(w), W, P_x : x \in S)$ taking values in S. Here W is a space of path functions w, and we denote the initial point by the subscript x in P_x. Precise definitions will be given in Section 2. The word 'Markov process' is used for time homogeneous Markov process in this paper. We define operators $G_\alpha : B(S) \to B(S)$ and $H_\alpha : B(\partial D) \to B(S)$ by

$$G_\alpha f(x) = E_x\Big[\int_0^\zeta e^{-\alpha t} f(x_t) dt\Big]$$

and

$$H_\alpha f(x) = E_x[e^{-\alpha\sigma} f(x_\sigma)] ,$$

where $\zeta = \zeta(w)$ is the lifetime, $\sigma = \sigma(w)$ is the first hitting time to ∂D, and E_x is the integration by P_x. We call G_α the α-order Green operator of X, and H_α the α-order hitting operator to ∂D of X. G_α [H_α] is an integral operator by a measure $G_\alpha(x, dy)$ [$H_\alpha(x, dy)$] on S [∂D], called the α-order Green measure [α-order hitting measure to ∂D]. Further, define $G_\alpha^{\min} : B(S) \to B(S)$ by

$$G_\alpha^{\min} f(x) = E_x\Big[\int_0^{\mathrm{Min}(\sigma,\zeta)} e^{-\alpha t} f(x_t) dt\Big] .$$

Then, $G_\alpha^{\min}$ is the α-order Green operator of a Markov process, which we call *the minimal part of X.* To say intuitively, we get the minimal part, killing x_t at the instant x_t reaches ∂D. Roughly speaking, the motion of X is determined by its minimal part and its behavior on the boundary. But, how can we characterize the behavior on the boundary? This is not simple, since the time spent by X on the boundary may have zero Lebesgue measure. We are concerned with this problem under some conditions.

Let m be a measure on S finite for any compact set, and let $m(\partial D) = 0$. m is fixed through this paper except in Section 6. We assume that the Markov process X satisfies Condition (A) stated in Section 2. Condition (A) requires, among others, that $G_\alpha(x, dy)$ is expressed by $g_\alpha(x, y) m(dy)$ for each x