J. Math. Soc. Japan
Vol. 17, No. 2, 1965

A criterion for the existence of a non-trivial partition of a finite group with applications to finite reflection groups

By Nagayoshi IWAHORI and Takeshi KONDO

(Received Nov. 24, 1964)

§ 0. Introduction

The purpose of this paper is to give a criterion for the existence of a non-trivial partition of a finite group G in terms of the existence of a certain permutation representation of G which we have considered in our previous papers [**6**] and [**7**]. We shall also give several applications of this criterion.

We refer to Baer [**1**], [**2**], [**3**]; Kegel [**8**], [**9**]; Kegel-Wall [**10**]; Suzuki [**11**], [**13**] as for basic concepts and theorems about the partitions of a group.

A group G was called of positive type in [**6**] if there exist a positive integer k and a G-space M (this means that G acts on M as a transformation group) with the following two properties:

(i) every element σ in $G-\{1\}$ has exactly k fixed points in M, and

(ii) no point in M is fixed by all elements in G. (We have called in [**6**] such a G-space M to be of type k. We shall also say that G is of type k on the G-space M.)

Now our criterion is stated as follows:

THEOREM 1. *A finite group G has a non-trivial partition if and only if G is of positive type.*

Although the proof of this theorem is quite elementary, it is divided into several steps and will be given in § 1.

In [**6**, Theorems IV and V], we have tried to distinguish the groups of positive type among Chevalley groups over a finite field. However our result there was not complete. Now by a profound result of Suzuki [**13**] and by the criterion above, this question is settled immediately. Let us state here Suzuki's theorem in a modified form for the convenience of the reader:

THEOREM 2. *Let G be a finite semi-simple group. (Recall that a finite group is called semi-simple if it has no nilpotent normal subgroups other than the unit group. Thus the semi-simplicity is equivalent to the non-existence of non-trivial abelian normal subgroups.) Then G is of positive type if and only if G is isomorphic with one of the following groups:*