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On the equivalence problems associated with a certain class of homogeneous spaces

By Noboru TANAKA

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Introduction

The present paper, first of all, introduces the notion of an l-system on which our theory is based. An l-system L is defined to be a system of a real semi-simple Lie algebra $\mathfrak{g}$ and three subalgebras of $\mathfrak{g}$ satisfying certain conditions (Definition 1.1). To every l-system L we associate a homogeneous space $M_L = G/G'$ of a Lie group G over a closed subgroup G' of G (see § 1). It is remarkable that the homogeneous space $M_L = G/G'$ is a prolongation of a compact Riemannian symmetric space in the following sense (cf. Proposition 3.2): A maximal compact subgroup K of G acts transitively on M_L, and the homogeneous space $M_L = K/K \cap G'$ is a compact Riemannian symmetric space. A recent work of T. Nagano [7] proves that, roughly speaking, any prolongation of a compact Riemannian symmetric space is locally isomorphic with a homogeneous space of the form $M_L = G/G'$.

Now let L be an l-system and let $M_L = G/G'$ be the corresponding homogeneous space. G being considered as a transformation group on M_L, the linear isotropy group $\tilde{G}$ of G at the origin o of M_L is a subgroup of the general linear group $GL(\mathfrak{m})$ of the tangent vector space $\mathfrak{m}$ to M_L at o. In this way, to every l-system L there corresponds a representation $(\tilde{G}, \mathfrak{m})$. Therefore there can be defined the notion of a $\tilde{G}$-structure: A $\tilde{G}$-structure on a manifold M is a principal fiber bundle $\tilde{P}$ over the base space M with structure group $\tilde{G}$ which is a subbundle of the bundle of frames of M (Definition 5.1).

The main purpose of the present paper is, for a given l-system L, to study conditions for the equivalence of two $\tilde{G}$-structures. Our main results (Theorems 9.3, 9.4, 10.1 and 10.2) may be stated as follows: Under general hypotheses on L, to every $\tilde{G}$-structure $\tilde{P}$ there is associated a system, called the normal connection of type (L), in such a way that the equivalence of two $\tilde{G}$-structures can be characterized. The normal connection of type (L) is a Cartan connection corresponding to the homogeneous space $M_L = G/G'$ and is found to be a generalization of the normal conformal connection. It should be here noted that there also exists the notion of the normal connection of