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Extension of certain subfields to coefficient fields in commutative algebras

By J. N. MORDESON and B. VINOGRADE*)

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Introduction.

Let A be a commutative algebra with identity over a subfield K. Let N be a maximal ideal of A and let g be the natural K-homomorphism of A onto A/N (K and gK identified). Denote A/N by F_0. Then, consistent with the usual meaning of the term coefficient field, we define a K-coefficient field as a subfield F of A such that $F \supseteqq K$ and $gF = F_0$.

The existence of coefficient fields for complete local algebras is assured by well known results [**3**, p. 106], but as simple examples show, the existence of K-coefficient fields is not a consequence. In Theorem 1, we give a necessary and sufficient condition for the stepwise extension of suitable subfields of A to K-coefficient fields when K has characteristic $p \neq 0$. These suitable subfields are situated in $A^{p^e} = \{a^{p^e} | a \in A\}$, e a positive integer, analogous to the way a K-coefficient field would be situated in A. This result applies of course to quasi-local algebras. In Theorem 2, we note an extension to the case of arbitrary characteristic of a result in [**2**] which can also be obtained by a modification of the proof of Corollary 2 in [**4**, p. 280], namely, the existence of a K-coefficient field when A is quasi-local, N is nil and F_0 has a separating transcendence basis over K. This theorem reduces the case of any quasi-local algebra with N nil to the case to which Theorem 1 applies.

1. By a counterimage $M \subseteqq A$ of a set $M_0 \subseteqq F_0$, we mean a set M such that $gM = M_0$ and $g|M$ is one-one. Unless otherwise specified, e always denotes a fixed positive integer. Let $M^{p^e} = \{m^{p^e} | m \in M\}$, and similarly for other prime powers of sets appearing hereafter. By the symbol $E(M)$ we mean the set of all polynomials in elements from M with coefficients from a field E.

LEMMA 1. *Suppose there exists a field $E \subseteqq A$ with the same identity as A such that $gE = F_0^{p^e}$. Then a counterimage $M \subseteqq A$ of a p-basis M_0 of F_0,* [**3**, p. 107], *is such that $M^{p^e} \subseteqq E$ if and only if $E(M)$ is a field. If such an M exists, $gE(M) = F_0$.*

PROOF. Suppose $M^{p^e} \subseteqq E$. Well order M and put $M_j = \{m_\alpha | \alpha < j\}$ for an ordinal j. Suppose $E(M_j)$ is a field for some ordinal j. Now m_j satisfies

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