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Sufficient conditions for p-valence of regular functions

By Noriyuki SONE

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§ 1. Introduction.

An interesting sufficient condition for univalence due to Umezawa [**18**, p. 213], [**16**, p. 191] and Kaplan [**5**, p. 173] has been generalized by Ogawa in his paper [**7**] as 'Main criterion' or as 'Theorem 2', while the last result has also been extended by Sakaguchi [**13**] as follows.

THEOREM A. *Let* $f(z)=z^p+\cdots$, $\varphi(z)$ *be regular in* $|z|\leqq r$ *and* $|z|<+\infty$ *respectively, and let* $f'(z)\neq 0$ *for* $0<|z|\leqq r$. *If neither* $f(z)$ *nor* $\varphi'(\log f(z))$ *vanishes on* $|z|=r$ *and the inequality*

$$\int_C d \arg d\varphi(\log f(z)) > -\pi$$

holds for any arc C *on* $|z|=r$, *then* $f(z)$ *is* p*-valent in* $|z|\leqq r$.

The purpose of this paper is to extend or improve the above results and some of other ones in [**6**], [**7**] and [**13**] by a systematic method. Some of our results may include, in a certain sense, a few new classes of uni- or multivalent functions.

§ 2. Fundamental propositions.

In this paper, we mainly consider the functions belonging to the class which is defined as follows.

DEFINITION 1. A function $f(z)$ is said to be a member of the class $\mathfrak{F}(p, D_z)$, where p is a positive integer and D_z is a simply connected closed domain whose boundary $\partial D_z\equiv C_z$ consists of a piecewise regular curve [**1**, p. 65] and whose interior contains the origin, if $f(z)$ is regular in D_z and has the expansion about the origin

$$f(z)=z^p+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\cdots,$$

and if $f(z)f'(z)\neq 0$ except at the origin in D_z.

Let C'_z denote any continuous, directed sub-arc of $C_z\equiv\partial D_z$, and let C'_w and C_w denote the images of C'_z and C_z by the mapping $w=f(z)$ respectively. The direction of C'_z is always generated, as usual, in the positive sense with respect