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On meromorphic and circumferentially mean univalent functions

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Introduction.

It is well known that the so-called one-quarter theorem plays an important role in the theory of regular and univalent functions in $|z|<1$. This theorem was extended to the case of circumferentially mean univalence (defined in § 1) by Hayman [6] and moreover to the case of areally mean univalence by Garabedian and Royden [5]. Their method was based on the fact that inner radius does not decrease by circular symmetrization (cf. [7]). On the other hand, corresponding to the one-quarter theorem, the following Montel-Bieberbach's theorem ([2], [3], [13], [14]) is well known in the case of meromorphic and univalent functions.

If $f(z)=z+a_2z^2+\cdots$ *is meromorphic and univalent in* $|z|<1$, *then at least one of the circles* $|w|<\delta$ *or* $|w|>\delta^{-1}$ $(\delta=\sqrt{5}-2)$ *is wholly covered by the image-domain under* $w=f(z)$.

In this paper we shall first prove a fundamental theorem on meromorphic and circumferentially mean univalent functions in $|z|<1$, by means of the fact that transfinite diameter does not increase by circular symmetrization and then generalized Montel-Bieberbach's theorem to the case of circumferentially mean univalence or p-valence.

Secondly we shall deal with values omitted by meromorphic and circumferentially mean univalent functions in $|z|<1$ also by means of the above mentioned property of transfinite diameter.

Thirdly we consider meromorphic and circumferentially mean univalent functions in $|z|<1$, whose Taylor expansions about the origin are given by $f(z)=z+a_2z^2+\cdots$ and whose poles are explicitly denoted by $z=z_\infty$, (as will be remarked in § 1, $f(z)$ has only one simple pole in $|z|<1$). By means of the pole $z=z_\infty$ we shall evaluate the values taken by $w=f(z)$ and its second Taylor coefficient a_2. Moreover a type of distortion theorem based on the pole $z=z_\infty$ will be derived.