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Semi-groups of operators in locally convex spaces

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The purpose of this paper is to extend some of the results in the theory of semi-groups of operators in Banach spaces to the case of locally convex topological vector spaces.

We consider a vector space $\mathfrak{X}$ with two locally convex topologies τ and σ which satisfy the conditions: (T 1) τ is stronger than σ; (T 2) τ has a base of neighborhoods of 0 composed of convex, circled and σ-closed sets; (T 3) $\mathfrak{X}_\tau$ is sequentially complete; (T 4) every continuous function $f(t)$ from $[0, 1]$ to $\mathfrak{X}_\sigma$ is Riemann integrable in σ. We shall call a family of linear operators $\{T_t\}_{t\geqq 0}$ in $\mathfrak{X}$ a (τ, σ) semi-group if it satisfies the conditions: (S 1) $T_0=I$ (identity); (S 2) $T_{t+s}=T_tT_s$; (S 3) $\{T_t\}$ is equicontinuous in $\mathfrak{L}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$; (S 4) T_t is, for every t, a σ-sequentially closed operator; (S 5) T_tx is a σ-continuous function for every x.

The well-known Hille-Yosida theory deals with the case $\tau=\sigma$ and when $\mathfrak{X}_\tau$ is a Banach space. The results have been generalized by Schwartz [8] when $\mathfrak{X}_\tau$ is a quasi-complete locally convex space. The theory in the case when $\mathfrak{X}$ is an adjoint space of a Banach space, τ is the strong topology and σ is the weak* topology is known as the theory of adjoint semi-groups by Feller [2] and Phillips [7].

In § 1 we give several sufficient conditions to assure the above assumptions. Especially it is shown that if $\mathfrak{X}_\tau$ is quasi-complete and if $\{T_t\}$ satisfies (S 1)—(S 3) and the condition that T_tx converges weakly to x as $t\to 0$ for every x, then $\{T_t\}$ is a (τ, τ) semi-group. § 2 is of preliminary nature.

The infinitesimal generators A_τ and A_σ are defined as usual by

$$A_\tau x=\tau\text{-}\lim_{t\to 0}\frac{1}{t}(T_t-I)x \quad \text{and} \quad A_\sigma x=\sigma\text{-}\lim_{t\to 0}\frac{1}{t}(T_t-I)x\,.$$

Thanks to the above assumptions, we can show that the Laplace transform

$$R(\lambda)x=\int_0^\infty e^{-\lambda t}T_txdt\,, \qquad \operatorname{Re}\lambda>0$$

is convergent as an improper σ-Riemann integral. $\{(\lambda R(\lambda))^m\}$ is equicontinuous in $\mathfrak{L}(\mathfrak{X}_\tau, \mathfrak{X}_\tau)$ with $\{T_t\}$ and $R(\lambda)$ is the resolvent of a τ-closed linear operator A, which we call the generator of $\{T_t\}$. We are mainly concerned in § 3