J. Math. Soc. Japan
Vol. 16, No. 3, 1964

On 1-cohomology groups of infinite dimensional representations of semisimple Lie algebras

By Akira HATTORI

(Received Feb. 22, 1964)

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field K of characteristic 0. Then all finite dimensional representations of $\mathfrak{g}$ are completely reducible. In homological language, this well-known theorem of Weyl is expressed by saying that $H^1(\mathfrak{g}, V)=0$ for every finite dimensional $\mathfrak{g}$-module V. Harish-Chandra [2] showed that the usual Cartan-Weyl theory is extendible in a large extent to some wide class of infinite dimensional representations. But the complete reducibility fails to hold for them. Indeed, simple examples show that $\mathrm{Ext}^1(K, V)=H^1(\mathfrak{g}, V)\neq 0$ for certain irreducible spaces V(see below). The purpose of the following lines is to determine the structure of $H^1(\mathfrak{g}, V)$ for irreducible spaces of that type.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We shall make use of the following simple exact sequence (established in more general setting in Hirata [4] and Hattori [3]):

(1) $$0\rightarrow H^1(\mathfrak{g}, \mathfrak{h}, V)\rightarrow H^1(\mathfrak{g}, V)\rightarrow H^1(\mathfrak{h}, V).$$

In the general setting, the relative cohomology group $H^1(\mathfrak{g}, \mathfrak{h}, V)$ is the one defined by Hochschild [5]. In the present case, $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$, and the relative cohomology groups coincide with those defined by Chevalley and Eilenberg [1] as is shown in [5].

The structure of $H^1(\mathfrak{h}, V)$ is quite simple. In general, we have

LEMMA 1. *Let $\mathfrak{h}$ be an abelian Lie algebra, and V^μ be an $\mathfrak{h}$-module such that $hv=\mu(h)v$ for every $h\in\mathfrak{h}$, $v\in V^\mu$, where μ is a linear form on $\mathfrak{h}$. Then we have, for $n=0, 1, 2, \cdots$,*

$$H^n(\mathfrak{h}, V^\mu)=\begin{cases}\mathrm{Hom}_K(E^n(\mathfrak{h}), V^\mu) & (\mu=0),\\ 0 & (\mu\neq 0),\end{cases}$$

where $E^n(\mathfrak{h})$ is the homogeneous component of degree n of the exterior algebra of $\mathfrak{h}$.

PROOF. An n-cochain $f\in C^n(\mathfrak{h}, V^\mu)$ is an n-cocycle if and only if

$$\mu(h_0)f(h_1, \cdots, h_n)-\mu(h_1)f(h_0, h_2, \cdots, h_n)+\cdots\pm\mu(h_n)f(h_0, \cdots, h_{n-1})=0$$