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On certain arithmetical Dirichlet series

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Introduction.

Let $\mathfrak{Q}(x)=\mathfrak{Q}(x_1,\cdots,x_r)$ be a *positive definite* quadratic form with rational integral coefficients. Any homogeneous polynomial $f(x)=f(x_1,\cdots,x_r)$ in r variables will be called a spherical function with respect to $\mathfrak{Q}$ if $\Delta f=0$, Δ being the Laplacian with respect to the metric $\mathfrak{Q}$. For each positive integer n, let G_n be the totality of all $r\times r$ matrices g with rational integral coordinates satisfying $\mathfrak{Q}(xg)=n\mathfrak{Q}(x)$.

We shall consider such spherical functions f as satisfying

(a) $f(xg)=f(x)$ for any $g\in G_1$

(b) for each $n=1,2,\cdots$, $\sum_{g\in G_n} f(xg)$ is a constant multiple of $f(x)$.

For each spherical function f satisfying (a) (b), we denote by $\tau_f(n)$ $(n=1,2,\cdots)$ the constant multiplicator in (b) divided by the cardinal number of G_1, and put

$$Z_f(s)=\sum_{n=1}^{\infty}\tau_f(n)n^{-s}.$$

Under a certain condition for $\mathfrak{Q}$, $Z_f(s)$ has an Euler-product expression, i.e. it is the product of

$$E_f^{(p)}(x)=\sum_{n=0}^{\infty}\tau_f(p^n)x^n;\quad x=p^{-s}$$

for all primes p, and it is often the case that $E_f^{(p)}(x)$ are rational functions of x. We are interested in the zeros and poles of such $E_f^{(p)}(x)$, since we expect "Riemann conjectures" about them. Many questions arise, but before attacking them, we wish to develop the arguments and find out our functions more explicitly for the simplest non-trivial cases. One of which is the case of $r=5$[1]. But the isomorphism between $O(5)$ and $Sp(4)$ suggests that we shall

1) A certain reformulation of several works of Eichler [1], [2], [3], [4] shows that in the case $r=3$, our $Z_f(s)$ are essentially the same as the Dirichlet series corresponding (by Mellin transformation) to modular forms, hence we cannot obtain anything new out of the case $r=3$. The case $r=4$ is reduced to that of $r=3$.