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On some modules in the theory of cyclotomic fields

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Let p be a fixed odd prime, and let $q_n = p^{n+1}$ for any integer $n \geqq 0$. Let F_n denote the cyclotomic field of q_n-th roots of unity over the rational field; and Φ_n, the local cyclotomic field of q_n-th roots of unity over the p-adic number field. The main purpose of the present paper is to introduce three compact modules $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ into the theory of cyclotomic fields F_n and Φ_n, $n \geqq 0$. They are defined as inverse limits of certain subgroups $\mathfrak{X}_n$, $\mathfrak{Y}_n$, and $\mathfrak{Z}_n$ respectively, of the additive group of Φ_n, $n \geqq 0$, and $\mathfrak{Y}$ is a submodule of $\mathfrak{X}$; $\mathfrak{Z}$, a submodule of $\mathfrak{Y}$. We shall determine the algebraic structure of $\mathfrak{X}/\mathfrak{Z}$ by direct computation, and we shall also show by class field theory how $\mathfrak{X}/\mathfrak{Y}$ and $\mathfrak{Y}/\mathfrak{Z}$ are related respectively to the ideal class groups and the unit groups of the fields F_n, $n \geqq 0$. Using these results, we shall then study the group-theoretical meaning of the classical class number formula for F_n as noted in a previous paper [**10**].

§1.

1.1. Let $\boldsymbol{Z}$, $\boldsymbol{Z}_p$, $\boldsymbol{Q}$ and $\boldsymbol{Q}_p$ denote the ring of rational integers, the ring of p-adic integers, the rational field, and the p-adic number field respectively. We shall fix an algebraic closure Ω of $\boldsymbol{Q}_p$, and consider all algebraic extensions of $\boldsymbol{Q}$ and $\boldsymbol{Q}_p$ as subfields of Ω.

Let F denote the union of all F_n, $n \geqq 0$; and Φ, the union of all Φ_n, $n \geqq 0$. Then both $F/\boldsymbol{Q}$ and $\Phi/\boldsymbol{Q}_p$ are abelian extensions, and their Galois groups are identified in a natural way. Put

$$G = G(F/\boldsymbol{Q}) = G(\Phi/\boldsymbol{Q}_p)^{1)},$$
$$G_n = G(F_n/\boldsymbol{Q}) = G(\Phi_n/\boldsymbol{Q}_p),$$
$$\Gamma = G(F/F_0) = G(\Phi/\Phi_0),$$
$$\Gamma_n = G(F/F_n) = G(\Phi/\Phi_n), \qquad n \geqq 0,$$

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1) $G(\ /\ )$ shall denote the Galois group of the Galois extension in the parenthesis.